PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 22:

Summary of mathematical methods

1. Fourier transforms
2. Fast Fourier transforms

6	Mon, 9/09/2013	Chap. 3	Calculus of variations -- continued	
7	Wed, 9/11/2013	Chap. 3	Calculus of variations applied to Lagrangians	#6
8	Fri, 9/13/2013	Chap. 3	Lagrangian mechanics	#7
9	Mon, 9/16/2013	Chap. 3 & 6	Lagrangian mechanics	#8
10	Wed, 9/18/2013	Chap. 3 & 6	Lagrangian mechanics	#9
11	Fri, 9/20/2013	Chap. 3 & 6	Lagrangian & Hamiltonian mechanics	#10
12	Mon, 9/23/2013	Chap. 3 & 6	Hamiltonian formalism	#11
13	Wed, 9/25/2013	Chap. 3 & 6	Hamiltonian formalism	#12
14	Fri, 9/27/2013	Chap. 3 & 6	Hamiltonian formalism	#13
15	Mon, 9/30/2013	Chap. 4	Small Oscillations	#14
16	Wed, 10/02/2013	Chap. 4	Small Oscillations	
17	Fri, 10/04/2013	Chap. 4	Small Oscillations	#15
18	Mon, 10/07/2013	Chap. 4 & 7	Small Oscillations and waves	#16
19	Wed, 10/09/2013	Chap. 7	Wave equation	
	Fri, 10/11/2013		No class (Fall Break)	
20	Mon, 10/14/2013	Chap. 7	Wave equation (Presentation topic due)	#17
21	Wed, 10/16/2013	Chap. 7	Mathematical methods	#18
22	Fri, 10/18/2013	Chap. 5	Mathematical methods	#19
23	Mon, 10/21/2013	Chap. 5	Rigid rotations	
24	Wed, 10/23/2013	Chap. 5	Rigid rotations	
25	Fri, 10/25/2013	Chap. 5	Rigid rotations	
	Mon, 10/28/2013	No class	Take-home exam	
	Wed, 10/30/2013	No class	Take-home exam	
	Fri, 11/01/2013	No class	Take-home exam	
26	Mon, 11/04/2013		Take-home exam due	

Use of Fourier transforms to solve wave equation

Wave equation: $\dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0$

Suppose $u(x,t) = e^{-i\omega t}\widetilde{F}(x,\omega)$ where $\widetilde{F}(x,\omega)$ satisfies the equation:

$$\frac{\partial^2 \widetilde{F}(x,\omega)}{\partial x^2} = -\frac{\omega^2}{c^2}\widetilde{F}(x,\omega) \equiv -k^2 \widetilde{F}(x,\omega)$$

Further assume that fixed boundary conditions apply: $0 \le x \le L$

with $\widetilde{F}(0,\omega) = 0$ and $\widetilde{F}(L,\omega) = 0$

For $n = 1,2,3\cdots$

$$\widetilde{F}_n(x,\omega) = \sin\left(\frac{n\pi x}{L}\right) \qquad k \to k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$$

$$u(x,t) = e^{-i\omega_n t}\sin(k_n x) = e^{-i\omega_n t}\frac{\left(e^{ik_n x} - e^{-ik_n x}\right)}{2i} = \frac{\left(e^{ik_n(x-ct)} - e^{-ik_n(x+ct)}\right)}{2i}$$

Use of Fourier transforms to solve wave equation -- continued

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Using superposition : Suppose $u(x,t) = \sum_n C_n e^{-i\omega_n t} \tilde{F}_n(x,\omega_n)$

$$\frac{\partial^2 \tilde{F}_n(x,\omega_n)}{\partial x^2} = -\frac{\omega_n^2}{c^2} \tilde{F}(x,\omega_n) \equiv -k_n^2 \, \tilde{F}(x,\omega_n)$$

For $\tilde{F}_n(x,\omega) = \sin\left(\frac{n\pi x}{L}\right)$ $\qquad k \to k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$

$$\Rightarrow u(x,t) = \sum_n C_n e^{-i\omega_n t} \sin(k_n x) = \sum_n \frac{C_n}{2i} e^{-i\omega_n t} \left(e^{ik_n x} - e^{-ik_n x}\right)$$

$$= \sum_n \frac{C_n}{2i} \left(e^{ik_n (x-ct)} - e^{-ik_n (x+ct)}\right) \equiv f(x-ct) + g(x+ct)$$

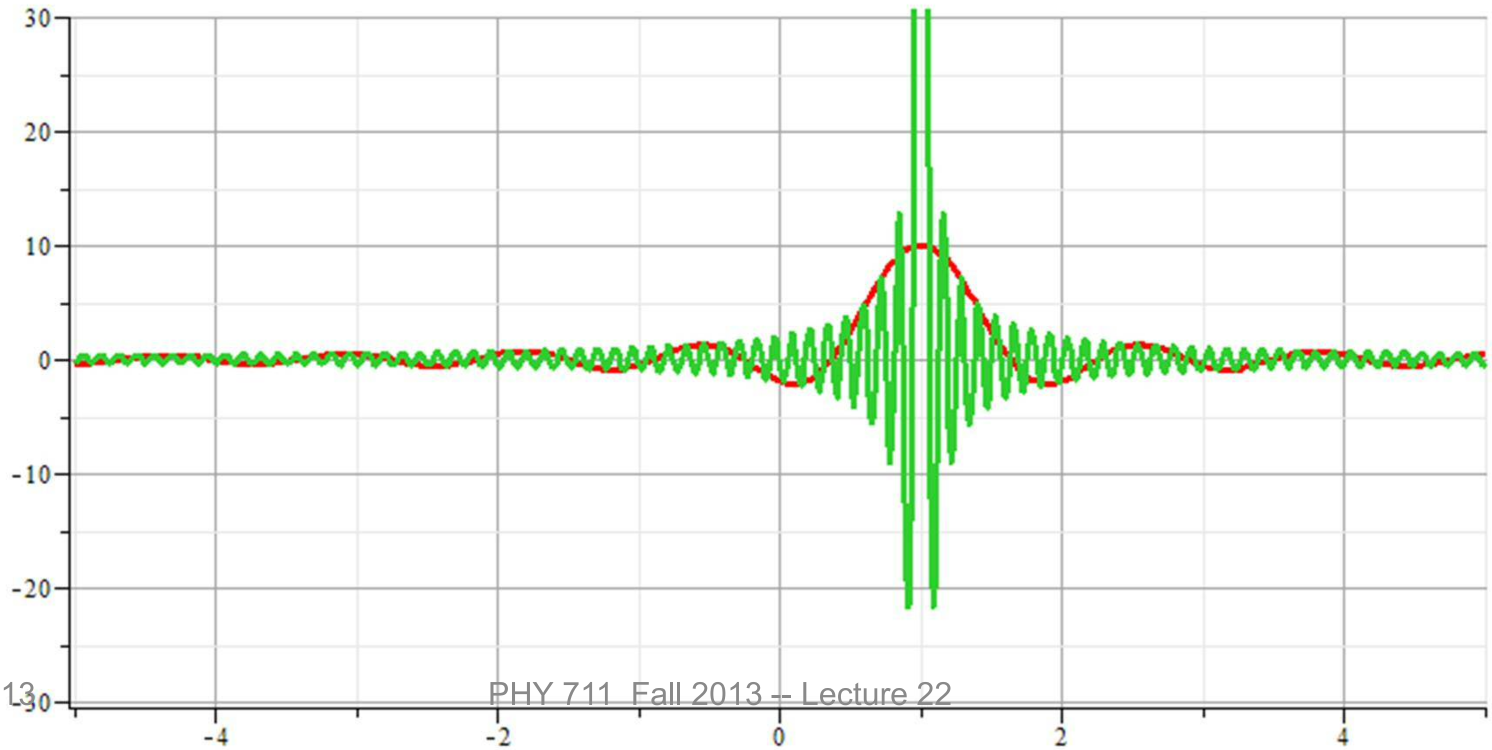
Fourier transforms

A useful identity

$$\int_{-\infty}^{\infty} dt\, e^{-i(\omega-\omega_0)t} = 2\pi\delta(\omega-\omega_0)$$

Note that

$$\int_{-T}^{T} dt\, e^{-i(\omega-\omega_0)t} = \frac{2\sin[(\omega-\omega_0)T]}{\omega-\omega_0}$$



Definition of Fourier Transform for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} d\omega \, F(\omega) \, e^{-i\omega t}$$

Backward transform:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, f(t) \, e^{i\omega t}$$

Check:

$$f(t) = \int_{-\infty}^{\infty} d\omega \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} dt' \, f(t') \, e^{i\omega t'} \right) e^{-i\omega t}$$

$$f(t) = \int_{-\infty}^{\infty} dt' \, f(t') \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \;\; e^{i\omega(t'-t)} \right) = \int_{-\infty}^{\infty} dt' \, f(t') \, \delta(t'-t)$$

Properties of Fourier transforms -- Parseval's theorem:

$$\int_{-\infty}^{\infty} dt \left(f(t)\right)^* f(t) = \int_{-\infty}^{\infty} d\omega \left(F(\omega)\right)^* F(\omega)$$

Fourier transform for periodic function:

Suppose $f(t+nT) = f(t)$ for any integer n

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t) e^{i\omega t} = \sum_{n=-\infty}^{\infty}\left(\int_0^T dt\, f(t) e^{i\omega(t+nT)}\right)$$

Note that:

$$\sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{\nu=-\infty}^{\infty} \delta(\omega - \nu\Omega), \quad \text{where} \quad \Omega \equiv \frac{2\pi}{T}$$

Some details :

$$\sum_{n=-M}^{M} e^{in\omega T} = \frac{\sin\left(\left(M+\frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}$$

$$\lim_{M\to\infty}\left(\frac{\sin\left(\left(M+\frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}\right) = 2\pi\sum_{\nu}\delta(\omega T - \nu\Omega T) = \frac{2\pi}{T}\sum_{\nu}\delta(\omega - \nu\Omega)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega\sum_{\nu=-\infty}^{\infty}\delta(\omega-\nu\Omega), \quad \text{where} \quad \Omega \equiv \frac{2\pi}{T}$$

$$\Rightarrow F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t) e^{i\omega t} = \sum_{\nu=-\infty}^{\infty}\Omega\delta(\omega-\nu\Omega)\left(\int_{0}^{T} dt\, f(t) e^{i\omega t}\right)$$

$$\equiv \sum_{\nu=-\infty}^{\infty}\Omega\delta(\omega-\nu\Omega)\overline{F}(\nu\Omega)$$

Fourier transform for periodic function – continued:

$$f(t) = \int_{-\infty}^{\infty} d\omega\, F(\omega)\, e^{-i\omega t} \quad \text{where} \quad f(t) = f(t+nT)$$

$$\Rightarrow F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t) e^{i\omega t} = \sum_{\nu=-\infty}^{\infty} \Omega\delta(\omega-\nu\Omega)\left(\int_0^T dt\, f(t) e^{i\omega t}\right)$$

$$\equiv \sum_{\nu=-\infty}^{\infty} \Omega\delta(\omega-\nu\Omega)\overline{F}(\nu\Omega)$$

$$\Omega \equiv \frac{2\pi}{T}$$

Thus, for a periodic function

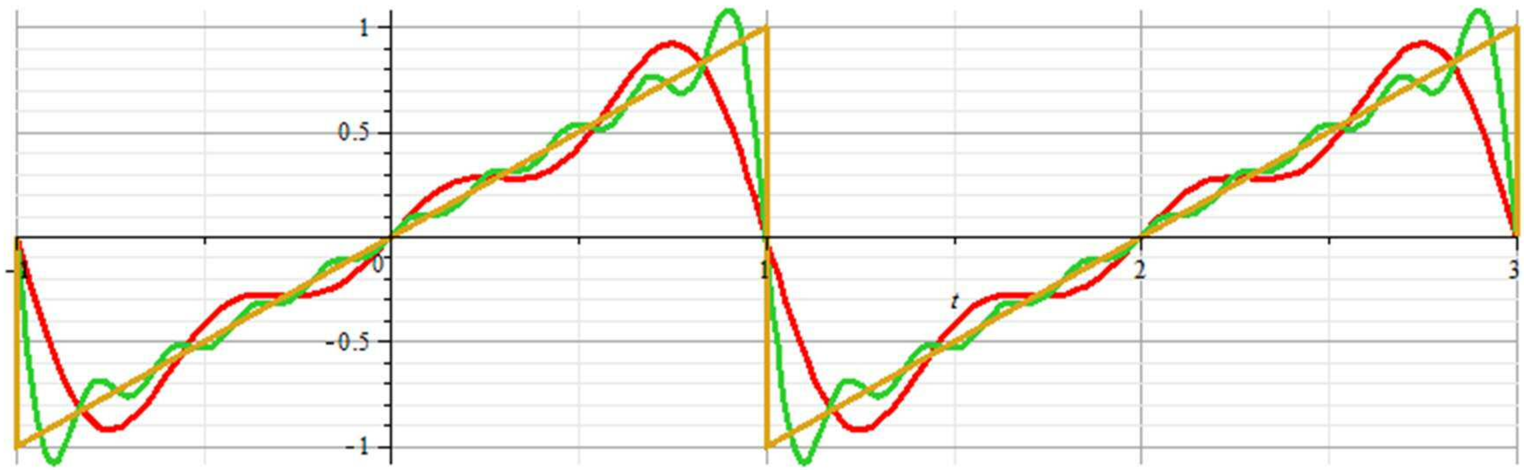
$$f(t) = \Omega \sum_{\nu=-\infty}^{\infty} \overline{F}(\nu\Omega) e^{-i\nu\Omega t}$$

Example:

Suppose: $f(t)=\dfrac{t-nT}{T}$ for $(n-1)T \le t \le (n+1)T$; $n=0,2,4,6\ldots$

Note, in this case the repeat period is $2T$ and the convenient sample time interval is $-T \le t \le T$.

$$\overline{F}(\nu\Omega)=\frac{2\pi}{2T}i\int_{-T}^{T}\frac{t}{T}\sin\left(\frac{\nu\pi t}{2T}\right)dt \qquad f(t)=\sum_{\nu=1}^{\infty}2\left|\overline{F}(\nu\Omega)\right|\sin\left(\frac{\nu\pi t}{2T}\right)$$



Thus, for a periodic function

$$f(t)=\sum_{\nu=-\infty}^{\infty}F(\nu\Omega)e^{-i\nu\Omega t}$$

Now suppose that the transformed function is bounded;

$$|F(\nu\Omega)|\le\varepsilon \quad \text{for} \quad |\nu|\ge N$$

Define a periodic transform function function

$$\widetilde{F}(\nu\Omega)\equiv\widetilde{F}(\nu\Omega+\nu'((2N+1)\Omega))$$

Effect on time domain :

$$f(t)=\sum_{\nu=-\infty}^{\infty}\widetilde{F}(\nu\Omega)e^{-i\nu\Omega t} \;=\frac{2\pi}{(2N+1)\Omega}\sum_{\nu=-N}^{N}\widetilde{F}(\nu\Omega)e^{-i\nu\Omega t}\sum_{\mu}\delta\left(t-\frac{\mu T}{2N+1}\right)$$

Doubly periodic functions

$$t \to \frac{\mu T}{2N+1}$$

$$\tilde{f}_{\mu} = \frac{1}{2N+1} \sum_{\nu=-N}^{N} \tilde{F}_{\nu} e^{-i2\pi\nu\mu/(2N+1)}$$

$$\tilde{F}_{\nu} = \sum_{\mu=-N}^{N} \tilde{f}_{\mu} e^{i2\pi\nu\mu/(2N+1)}$$

More convenient notation

$$2N+1 \rightarrow M$$

$$\tilde{f}_\mu = \frac{1}{M}\sum_{\nu=0}^{M-1}\tilde{F}_\nu e^{-i2\pi\nu\mu/M}$$

$$\tilde{F}_\nu = \sum_{\mu=0}^{M}\tilde{f}_\mu e^{i2\pi\nu\mu/M}$$

Note that for $W = e^{i2\pi/M}$

$$\tilde{F}_0 = \tilde{f}_0 W^0 + \tilde{f}_1 W^0 + \tilde{f}_2 W^0 + \tilde{f}_3 W^0 + \cdots$$

$$\tilde{F}_1 = \tilde{f}_0 W^0 + \tilde{f}_1 W^1 + \tilde{f}_2 W^2 + \tilde{f}_3 W^3 + \cdots$$

$$\tilde{F}_2 = \tilde{f}_0 W^0 + \tilde{f}_1 W^2 + \tilde{f}_2 W^4 + \tilde{f}_3 W^6 + \cdots$$

Note that for $W = e^{i2\pi/M}$

$$\tilde{F}_0 = \tilde{f}_0 W^0 + \tilde{f}_1 W^0 + \tilde{f}_2 W^0 + \tilde{f}_3 W^0 + \cdots$$

$$\tilde{F}_1 = \tilde{f}_0 W^0 + \tilde{f}_1 W^1 + \tilde{f}_2 W^2 + \tilde{f}_3 W^3 + \cdots$$

$$\tilde{F}_2 = \tilde{f}_0 W^0 + \tilde{f}_1 W^2 + \tilde{f}_2 W^4 + \tilde{f}_3 W^6 + \cdots$$

However, $W^M = \left(e^{i2\pi/M}\right)^M = 1$

and $W^{M/2} = \left(e^{i2\pi/M}\right)^{M/2} = -1$

Cooley-Tukey algorithm: J. W. Cooley and J. W. Tukey, "An algorithm for machine calculation of complex Fourier series" Math. Computation 19, 297-301 (1965)