PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 20:

Summary of mathematical methods

1. Sturm-Liouville equations
2. Green's function methods
3. Laplace transform
4. Contour integration

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/28/2013	Chap. 1	Review of basic principles;Scattering theory	#1
2	Fri, 8/30/2013	Chap. 1	Scattering theory continued	#2
3	Mon, 9/02/2013	Chap. 1	Scattering theory continued	#3
4	Wed, 9/04/2013	Chap. 2	Accelerated Coordinate Systems	#4
5	Fri, 9/06/2013	Chap. 3	Calculus of variations	#5
6	Mon, 9/09/2013	Chap. 3	Calculus of variations -- continued	
7	Wed, 9/11/2013	Chap. 3	Calculus of variations applied to Lagrangians	#6
8	Fri, 9/13/2013	Chap. 3	Lagrangian mechanics	#7
9	Mon, 9/16/2013	Chap. 3 & 6	Lagrangian mechanics	#8
10	Wed, 9/18/2013	Chap. 3 & 6	Lagrangian mechanics	#9
11	Fri, 9/20/2013	Chap. 3 & 6	Lagrangian & Hamiltonian mechanics	#10
12	Mon, 9/23/2013	Chap. 3 & 6	Hamiltonian formalism	#11
13	Wed, 9/25/2013	Chap. 3 & 6	Hamiltonian formalism	#12
14	Fri, 9/27/2013	Chap. 3 & 6	Hamiltonian formalism	#13
15	Mon, 9/30/2013	Chap. 4	Small Oscillations	#14
16	Wed, 10/02/2013	Chap. 4	Small Oscillations	
17	Fri, 10/04/2013	Chap. 4	Small Oscillations	#15
18	Mon, 10/07/2013	Chap. 4 & 7	Small Oscillations and waves	#16
19	Wed, 10/09/2013	Chap. 7	Wave equation	
	Fri, 10/11/2013		No class (Fall Break)	
20	Mon, 10/14/2013	Chap. 7	Wave equation (Presentation topic due)	#17

Sturm-Liouville equation:

Homogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi_0(x)=0$

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi(x)=F(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Example: $\tau(x)=1;\quad \sigma(x)=1;\quad v(x)=0;\quad a=0$ and $b=L$

$$\lambda=1;\qquad F(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

Inhomogenous equation:

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

Eigenvalue equation :

$$\left(-\frac{d^2}{dx^2}\right)f_n(x) = \lambda_n f_n(x)$$

Eigenfunctions

$$f_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

Eigenvalues :

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Completeness of eigenfunctions :

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n} = \delta(x-x')$$

In this example :

$$\frac{2}{L}\sum_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right) = \delta(x-x')$$

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x)\right)G_\lambda(x,x') = \delta(x-x')$$

Green's function for the example :

$$G(x,x') = \sum_n \frac{f_n(x)f_n(x')/N_n}{\lambda_n - \lambda} = \frac{2}{L}\sum_n \frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\left(\frac{n\pi}{L}\right)^2 - 1}$$

Using Green's function to solve inhomogenous equation :

$$\left(-\frac{d^2}{dx^2} - 1\right)\phi(x) = F_0 \sin\left(\frac{\pi x}{L}\right)$$

$$\phi(x) = \phi_0(x) + \int_0^L G(x,x')F_0 \sin\left(\frac{\pi x'}{L}\right)dx'$$

$$= \phi_0(x) + \frac{2}{L}\sum_n \left[\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2 - 1}\int_0^L \sin\left(\frac{n\pi x'}{L}\right)F_0 \sin\left(\frac{\pi x'}{L}\right)dx'\right]$$

$$= \phi_0(x) + \frac{F_0}{\left(\frac{\pi}{L}\right)^2 - 1}\sin\left(\frac{\pi x}{L}\right)$$

Alternate Green's function method:

$$G(x,x') = \frac{1}{W} g_a(x_<) g_B(x_>)$$

$$\left(-\frac{d^2}{dx^2} - 1\right) g_i(x) = 0 \qquad \Rightarrow g_a(x) = \sin(x); \quad g_b(x) = \sin(L-x);$$

$$W = g_b(x)\frac{dg_a(x)}{dx} - g_a(x)\frac{dg_b(x)}{dx} = \sin(L-x)\cos(x) + \sin(x)\cos(L-x)$$

$$= \sin(L)$$

$$\phi(x) = \phi_0(x) + \frac{\sin(L-x)}{\sin(L)} \int_0^x \sin(x') F_0 \sin\left(\frac{\pi x'}{L}\right) dx'$$

$$+ \frac{\sin(x)}{\sin(L)} \int_x^L \sin(L-x') F_0 \sin\left(\frac{\pi x'}{L}\right) dx'$$

$$\phi(x) = \phi_0(x) + \frac{F_0}{\left(\frac{\pi}{L}\right)^2 - 1} \sin\left(\frac{\pi x}{L}\right)$$

Laplace transforms

Laplace transforms can be used to solve initial value problems. The Laplace transform of a function $\phi(x)$ is defined as

$$\mathcal{L}_\phi(p) \equiv \int_0^\infty e^{-px} \phi(x) dx. \qquad (24)$$

Assuming that $\phi(x)$ is well-behaved in the interval $0 \le x \le \infty$, the following properties are useful:

$$\mathcal{L}_{d\phi/dx}(p) = -\phi(0) + p\mathcal{L}_\phi(p), \qquad (25)$$

and

$$\mathcal{L}_{d^2\phi/dx^2}(p) = -\frac{d\phi(0)}{dx} - p\phi(0) + p^2 \mathcal{L}_\phi(p). \qquad (26)$$

These identities allow us to turn a differential equation for $\phi(x)$ into an algebraic equation for $\mathcal{L}_\phi(p)$. We then need to perform an inverse Laplace transform to find $\phi(x)$.
For illustration, we will consider a simple example with $\tau(x) = 1$, $\sigma(x) = 1$, $\lambda = 0$. The differential equation then becomes

$$-\frac{d^2\phi(x)}{dx^2} = F(x), \qquad (27)$$

where we will take the initial conditions to be $\phi(0) = 0$ and $d\phi(0)/dx = 0$. For our example, we will also take $F(x) = F_0 e^{-\gamma x}$. Multiplying, both sides of the equation by e^{-px} and integrating $0 \le x \le \infty$, we find

$$\mathcal{L}_\phi(p) = -\frac{F_0}{p^2(\gamma + p)}. \qquad (28)$$

In general the inverse Laplace transform involves performing a contour integral, but we can use the following simple relations

$$\mathcal{L}_1 = \int_0^\infty e^{-px} dx = \frac{1}{p}. \quad (29)$$

$$\mathcal{L}_x = \int_0^\infty x e^{-px} dx = \frac{1}{p^2}. \quad (30)$$

$$\mathcal{L}_{e^{-\gamma x}} = \int_0^\infty e^{-\gamma x} e^{-px} dx = \frac{1}{p+\gamma}. \quad (31)$$

Noting that

$$-\frac{F_0}{p^2(\gamma+p)} = -\frac{F_0}{\gamma^2}\left(\frac{1}{\gamma+p} - \frac{1}{p} + \frac{\gamma}{p^2}\right), \quad (32)$$

we see that the inverse Laplace transform gives us

$$\phi(x) = \frac{F_0}{\gamma^2}\left(1 - e^{-\gamma x} - \gamma x\right). \quad (33)$$

We can check that this a solution to the differential equation

$$-\frac{d^2\phi}{dx^2} = F_0 e^{-\gamma x} \qquad \text{for} \quad \phi(0) = 0 \qquad \text{and} \quad \frac{d\phi}{dx}(0) = 0$$

Using Laplace transforms to solve equation :

$$\left(-\frac{d^2}{dx^2} - 1\right)\phi(x) = F_0 \sin\left(\frac{\pi x}{L}\right) \qquad \text{with} \quad \phi(0) = 0, \ \frac{d\phi(0)}{dx} = 0$$

$$\mathcal{L}_\phi(p) = -\left(\frac{\pi}{L}\right)\frac{F_0}{\left(p^2+1\right)\left(p^2 + \left(\frac{\pi}{L}\right)^2\right)}$$

$$= -F_0\left(\frac{\pi/L}{(\pi/L)^2 - 1}\right)\left(\frac{1}{p^2+1} - \frac{1}{p^2 + \left(\frac{\pi}{L}\right)^2}\right)$$

Note that : $\int_0^\infty \sin(at) e^{-pt} dt = \frac{a}{a^2 + p^2}$

$$\Rightarrow \phi(x) = \frac{F_0}{(\pi/L)^2 - 1}\left(\sin\left(\frac{\pi x}{L}\right) - \frac{\pi}{L}\sin(x)\right)$$

Inverse Laplace transform :

$$\mathcal{L}_\phi(p) = \int_0^\infty e^{-pt}\phi(t) dt$$

$$\phi(t) = \frac{1}{2\pi i}\int_{\lambda - i\infty}^{\lambda + i\infty} e^{pt}\mathcal{L}_\phi(p) dp$$

Check : $\frac{1}{2\pi i}\int_{\lambda - i\infty}^{\lambda + i\infty} e^{pt}\mathcal{L}_\phi(p) dt = \frac{1}{2\pi i}\int_{\lambda - i\infty}^{\lambda + i\infty} e^{pt} dp \int_0^\infty e^{-pu}\phi(u) du$

$$\frac{1}{2\pi i}\int_0^\infty \phi(u) du \int_{\lambda - i\infty}^{\lambda + i\infty} e^{p(t-u)} dp = \frac{1}{2\pi i}\int_0^\infty \phi(u) du \int_{-\infty}^{\infty} e^{\lambda(t-u)} e^{is(t-u)} i ds$$

$$= \frac{1}{2\pi i}\int_0^\infty \phi(u) du \left(e^{\lambda(t-u)} 2\pi i \, \delta(t-u)\right)$$

$$= \begin{cases} \phi(t) & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$