

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF Olin 103**

Plan for Lecture 12:

Continue reading Chapter 3 & 6

- 1. Hamiltonian formalism**
- 2. Phase space**
- 3. Liouville's theorem**

9/22/2013

PHY 711 Fall 2013 – Lecture 12

1

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

Date	F&W Reading	Topic	Assignment
1 Wed, 8/28/2013	Chap. 1	Review of basic principles, Scattering theory	#1
2 Fri, 8/30/2013	Chap. 1	Scattering theory continued	#2
3 Mon, 9/02/2013	Chap. 1	Scattering theory continued	#3
4 Wed, 9/04/2013	Chap. 2	Accelerated Coordinate Systems	#4
5 Fri, 9/06/2013	Chap. 3	Calculus of variations	#5
6 Mon, 9/09/2013	Chap. 3	Calculus of variations -- continued	
7 Wed, 9/11/2013	Chap. 3	Calculus of variations applied to Lagrangians	#6
8 Fri, 9/13/2013	Chap. 3	Lagrangian mechanics	#7
9 Mon, 9/16/2013	Chap. 3 & 6	Lagrangian mechanics	#8
10 Wed, 9/18/2013	Chap. 3 & 6	Lagrangian mechanics	#9
11 Fri, 9/20/2013	Chap. 3 & 6	Lagrangian & Hamiltonian mechanics	#10
12 Mon, 9/23/2013	Chap. 3 & 6	Hamiltonian formalism	#11

9/22/2013

PHY 711 Fall 2013 – Lecture 12

2

Hamiltonian formalism

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

Canonical equations of motion

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma}$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

9/22/2013

PHY 711 Fall 2013 – Lecture 12

3

Hamiltonian formalism and time evolution:

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma}$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

$$\frac{dH}{dt} = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

For an arbitrary function : $F = F(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$\frac{dF}{dt} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial F}{\partial t} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \frac{\partial H}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma} \frac{\partial H}{\partial q_\sigma} \right) + \frac{\partial F}{\partial t}$$

9/22/2013 PHY 711 Fall 2013 – Lecture 12 4

Hamiltonian formalism and time evolution:

Poisson brackets:

For an arbitrary function : $F = F(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$\frac{dF}{dt} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial F}{\partial t} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \frac{\partial H}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma} \frac{\partial H}{\partial q_\sigma} \right) + \frac{\partial F}{\partial t}$$

Define :

$$[F, G]_{PB} \equiv \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma} \frac{\partial G}{\partial q_\sigma} \right) = -[G, F]_{PB}$$

So that : $\frac{dF}{dt} = [F, H]_{PB} + \frac{\partial F}{\partial t}$

9/22/2013 PHY 711 Fall 2013 – Lecture 12 5

Poisson brackets -- continued:

$$[F, G]_{PB} \equiv \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma} \frac{\partial G}{\partial q_\sigma} \right) = -[G, F]_{PB}$$

Examples :

$$[x, x]_{PB} = 0 \quad [x, p_x]_{PB} = 1 \quad [x, p_y]_{PB} = 0$$

$$[L_x, L_y]_{PB} = L_z$$

Liouville theorem

Let D \equiv density of particles in phase space :

$$\frac{dD}{dt} = [D, H]_{PB} + \frac{\partial D}{\partial t} = 0$$

9/22/2013 PHY 711 Fall 2013 – Lecture 12 6

Phase space

Phase space is defined as the set of all coordinates and momenta of a system :

$$(\{q_\sigma(t)\}, \{p_\sigma(t)\})$$

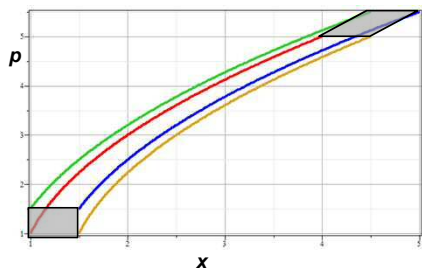
For a d dimensional system with N particles, the phase space corresponds to $2dN$ degrees of freedom.

9/22/2013

PHY 711 Fall 2013 – Lecture 12

7

Phase space diagram for one-dimensional motion due to constant force

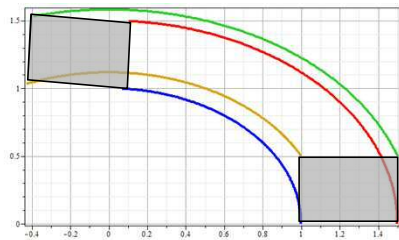


9/22/2013

PHY 711 Fall 2013 – Lecture 12

8

Phase space diagram for one-dimensional motion due to spring force



9/22/2013

PHY 711 Fall 2013 – Lecture 12

9

Liouville's Theorem (1838)

The density of representative points in phase space corresponding to the motion of a system of particles remains constant during the motion.

Denote the density of particles in phase space: $D = D(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$\frac{dD}{dt} = \sum_\sigma \left(\frac{\partial D}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial D}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial D}{\partial t}$$

According to Liouville's theorem: $\frac{dD}{dt} = 0$

9/22/2013 PHY 711 Fall 2013 – Lecture 12 10

Liouville's theorem

9/22/2013 PHY 711 Fall 2013 – Lecture 12 11

Liouville's theorem -- continued

$\frac{\partial D}{\partial t} \Rightarrow$ time rate of change of particles within volume
 = time rate of particle entering minus particles leaving
 $= -\frac{\partial D}{\partial x} \dot{x} - \frac{\partial D}{\partial p} \dot{p}$

9/22/2013 PHY 711 Fall 2013 – Lecture 12 12

Liouville's theorem -- continued

$$\frac{\partial D}{\partial t} = -\frac{\partial D}{\partial x} \dot{x} - \frac{\partial D}{\partial p} \dot{p}$$

$$\frac{\partial D}{\partial t} + \frac{\partial D}{\partial x} \dot{x} + \frac{\partial D}{\partial p} \dot{p} = 0 = \frac{dD}{dt}$$

9/22/2013 PHY 711 Fall 2013 - Lecture 12 13

$$\frac{dD}{dt} = 0$$

Importance of Liouville's theorem to statistical mechanical analysis:

In statistical mechanics, we need to evaluate the probability of various configurations of particles. The fact that the density of particles in phase space is constant in time, implies that each point in phase space is equally probable and that the time average of the evolution of a system can be determined by an average of the system over phase space volume.

9/22/2013 PHY 711 Fall 2013 - Lecture 12 14

Virial theorem (Clausius ~ 1860)

$$2\langle T \rangle = -\left\langle \sum_{\sigma} \mathbf{F}_{\sigma} \cdot \mathbf{r}_{\sigma} \right\rangle$$

Proof:

Define: $A \equiv \sum_{\sigma} \mathbf{p}_{\sigma} \cdot \mathbf{r}_{\sigma}$

$$\frac{dA}{dt} = \sum_{\sigma} (\dot{\mathbf{p}}_{\sigma} \cdot \mathbf{r}_{\sigma} + \mathbf{p}_{\sigma} \cdot \dot{\mathbf{r}}_{\sigma}) = \sum_{\sigma} \mathbf{F}_{\sigma} \cdot \mathbf{r} + 2T$$

$$\left\langle \frac{dA}{dt} \right\rangle = \left\langle \sum_{\sigma} \mathbf{F}_{\sigma} \cdot \mathbf{r} \right\rangle + 2\langle T \rangle$$

$$\left\langle \frac{dA}{dt} \right\rangle = \frac{1}{\tau} \int_0^{\tau} \frac{dA(t)}{dt} dt = \frac{A(\tau) - A(0)}{\tau} \Rightarrow 0$$

9/22/2013 PHY 711 Fall 2013 - Lecture 12 15
