PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 11:

Continue reading Chapter 3 & 6

1. **Constructing the Hamiltonian**
2. **Hamilton's canonical equation**
3. **Examples**

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/28/2013	Chap. 1	Review of basic principles;Scattering theory	#1
2	Fri, 8/30/2013	Chap. 1	Scattering theory continued	#2
3	Mon, 9/02/2013	Chap. 1	Scattering theory continued	#3
4	Wed, 9/04/2013	Chap. 2	Accelerated Coordinate Systems	#4
5	Fri, 9/06/2013	Chap. 3	Calculus of variations	#5
6	Mon, 9/09/2013	Chap. 3	Calculus of variations -- continued	
7	Wed, 9/11/2013	Chap. 3	Calculus of variations applied to Lagrangians	#6
8	Fri, 9/13/2013	Chap. 3	Lagrangian mechanics	#7
9	Mon, 9/16/2013	Chap. 3 & 6	Lagrangian mechanics	#8
10	Wed, 9/18/2013	Chap. 3 & 6	Lagrangian mechanics	#9
11	Fri, 9/20/2013	Chap. 3 & 6	Lagrangian & Hamiltonian mechanics	#10

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

Hamiltonian picture – continued

$$H = H(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \qquad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \qquad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

Direct application of Hamiltonian's principle using the Hamiltonian function --

Generalized coordinates: $q_\sigma(\{x_i\})$

Define -- Lagrangian: $L \equiv T - U$

$$L = L(\{q_\sigma\},\{\dot{q}_\sigma\},t)$$

$$\Rightarrow \text{Minimization integral:} \quad S = \int_{t_i}^{t_f} L(\{q_\sigma\},\{\dot{q}_\sigma\},t)\,dt$$

Expressed in terms of Hamiltonian:

$$H = H(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \Rightarrow L = \sum_\sigma \dot{q}_\sigma p_\sigma - H(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$$

Hamilton's principle continued:

Minimization integral:

$$S = \int_{t_i}^{t_f} \left(\sum_\sigma \dot{q}_\sigma p_\sigma - H(\{q_\sigma(t)\},\{p_\sigma(t)\},t) \right) dt$$

$$\delta S = \int_{t_i}^{t_f} \left(\sum_\sigma \left(\dot{q}_\sigma \delta p_\sigma + \delta\dot{q}_\sigma p_\sigma - \frac{\partial H}{\partial q_\sigma}\delta q_\sigma - \frac{\partial H}{\partial p_\sigma}\delta p_\sigma \right) \right) dt = 0$$

$$\Rightarrow \boxed{\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}}$$

$$\Rightarrow \boxed{\dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma}}$$

Canonical equations

Detail:

$$\int_{t_i}^{t_f} \left(\sum_\sigma (\delta\dot{q}_\sigma p_\sigma) \right) dt = \int_{t_i}^{t_f} \left(\sum_\sigma \left(\frac{d(\delta q_\sigma p_\sigma)}{dt} - \delta q_\sigma \dot{p}_\sigma \right) \right) dt = \overset{0}{\cancel{\sum_\sigma \delta q_\sigma p_\sigma \Big|_{t_i}^{t_f}}} - \int_{t_i}^{t_f} \left(\sum_\sigma (\delta q_\sigma \dot{p}_\sigma) \right) dt$$

Constants of the motion in Hamiltonian formalism

$$H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \Rightarrow \text{constant } \; q_\sigma \; \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \qquad \Rightarrow \text{constant } \; p_\sigma \; \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

$$\Rightarrow \text{constant } \; H \; \text{ if } \; \frac{\partial H}{\partial t} = 0$$

Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function : $L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$
2. Compute generalized momenta : $p_\sigma \equiv \dfrac{\partial L}{\partial \dot{q}_\sigma}$
3. Construct Hamiltonian expression : $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$
4. Form Hamiltonian function : $H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$
5. Analyze canonical equations of motion :

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

Example 1 : one - dimensional potential :

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)$$

$$p_x = m\dot{x} \qquad p_y = m\dot{y} \qquad p_z = m\dot{z}$$

$$H = m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2 - \left(\tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)\right)$$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + V(z)$$

Constants : p_x, p_y, H

Equations of motion : $\dfrac{dz}{dt} = \dfrac{\partial H}{\partial p_z} = \dfrac{p_z}{m} \qquad \dfrac{dp_z}{dt} = -\dfrac{dV}{dz}$

Example 2 : Motion in a central potential

$$L = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r)$$

$$p_r = m\dot{r} \qquad p_\phi = mr^2\dot{\phi}$$

$$H = m\dot{r}^2 + mr^2\dot{\phi}^2 - \left(\tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r)\right)$$

$$= \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + V(r)$$

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r)$$

Constants : p_ϕ, H

Equations of motion :

$$\frac{dr}{dt} = \frac{p_r}{m} \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{mr^3} - \frac{\partial V}{\partial r}$$

Other examples

Lagrangian for symmetric top with Euler angles α, β, γ :

$$L = L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \tfrac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2\right) + \tfrac{1}{2}I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2 - Mgh\cos\beta$$

$$p_\alpha = I_1\dot{\alpha}\sin^2\beta + I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)\cos\beta$$

$$p_\beta = I_1\dot{\beta}$$

$$p_\gamma = I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)$$

$$H = \tfrac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2\right) + \tfrac{1}{2}I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2 + Mgh\cos\beta$$

$$H = \frac{\left(p_\alpha - p_\gamma\cos\beta\right)^2}{2I_1\sin^2\beta} + \frac{p_\beta^2}{2I_1} + \frac{p_\gamma^2}{2I_3} + Mgh\cos\beta$$

Constants : p_α, p_γ, H

Other examples

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c}B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$p_x = m\dot{x} - \frac{q}{2c}B_0 y$$

$$p_y = m\dot{y} + \frac{q}{2c}B_0 x$$

$$p_z = m\dot{z}$$

$$H = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$H = \frac{\left(p_x + \frac{q}{2c}B_0 y\right)^2}{2m} + \frac{\left(p_y - \frac{q}{2c}B_0 x\right)^2}{2m} + \frac{p_z^2}{2m}$$

Constants : p_z, H

Canonical equations of motion for constant magnetic field:

$$H = \frac{\left(p_x + \frac{q}{2c}B_0 y\right)^2}{2m} + \frac{\left(p_y - \frac{q}{2c}B_0 x\right)^2}{2m} + \frac{p_z^2}{2m}$$

$$\text{Constants : } \quad p_z, H$$

$$\frac{dx}{dt} = \frac{p_x + \frac{q}{2c}B_0 y}{m} \qquad \frac{dy}{dt} = \frac{p_y - \frac{q}{2c}B_0 x}{m}$$

$$\frac{dp_x}{dt} = -\frac{\partial H}{\partial x} = \frac{qB_0}{2mc}\left(p_y - \frac{q}{2c}B_0 x\right)$$

$$\frac{dp_y}{dt} = -\frac{\partial H}{\partial y} = -\frac{qB_0}{2mc}\left(p_x + \frac{q}{2c}B_0 y\right)$$

Canonical equations of motion for constant magnetic field -- continued:

$$\frac{dx}{dt} = \frac{p_x + \frac{q}{2c}B_0 y}{m} \qquad \frac{dy}{dt} = \frac{p_y - \frac{q}{2c}B_0 x}{m}$$

$$\frac{dp_x}{dt} = \frac{qB_0}{2mc}\left(p_y - \frac{q}{2c}B_0 x\right) = \frac{qB_0}{2c}\frac{dy}{dt}$$

$$\frac{dp_y}{dt} = -\frac{qB_0}{2mc}\left(p_x + \frac{q}{2c}B_0 y\right) = -\frac{qB_0}{2c}\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{\dot{p}_x}{m} + \frac{q}{2mc}B_0\dot{y} = \frac{qB_0}{mc}\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} = \frac{\dot{p}_y}{m} - \frac{q}{2mc}B_0\dot{x} = -\frac{qB_0}{mc}\frac{dx}{dt}$$

Poisson brackets:
Recall:

$$H = H(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \quad \Rightarrow \text{constant } q_\sigma \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \quad \Rightarrow \text{constant } p_\sigma \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma\left(\frac{\partial H}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma}\dot{p}_\sigma\right) + \frac{\partial H}{\partial t}$$

Similarly for an arbitrary function : $F = F(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$

$$\frac{dF}{dt} = \sum_\sigma\left(\frac{\partial F}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma}\dot{p}_\sigma\right) + \frac{\partial F}{\partial t} = \sum_\sigma\left(\frac{\partial F}{\partial q_\sigma}\frac{\partial H}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial H}{\partial q_\sigma}\right) + \frac{\partial F}{\partial t}$$

Poisson brackets -- continued:

For an arbitrary function : $F = F(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$

$$\frac{dF}{dt} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial F}{\partial t} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\frac{\partial H}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial H}{\partial q_\sigma} \right) + \frac{\partial F}{\partial t}$$

Define :

$$[F,G]_{PB} \equiv \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial G}{\partial q_\sigma} \right) = -[G,F]_{PB}$$

So that : $\frac{dF}{dt} = [F,H]_{PB} + \frac{\partial F}{\partial t}$

Poisson brackets -- continued:

$$[F,G]_{PB} \equiv \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial G}{\partial q_\sigma} \right) = -[G,F]_{PB}$$

Examples :

$[x,x]_{PB} = 0 \qquad [x,p_x]_{PB} = 1 \qquad [x,p_y]_{PB} = 0$

Liouville theorem

Let $D \equiv$ density of particles in phase space :

$$\frac{dD}{dt} = [D,H]_{PB} + \frac{\partial D}{\partial t} = 0$$