Basic assumptions*

We assume that we have in incompressible fluid ($\rho =$ constant) a velocity potential of the form $\Phi(x, z, t)$, where

$$\mathbf{v}(x, z, t) = -\nabla\Phi(x, z, t). \tag{1}$$

The surface of the fluid is described by $z = h + \zeta(x, t)$. It is assumed that the fluid is contained in a structure (lake, river, swimming pool, etc.) with a structureless bottom defined by the $z = 0$ plane and filled to an equilibrium height of $z = h$. We assume that we have in incompressible fluid ($\rho =$ constant) a velocity potential of the form $\Phi(x, z, t)$, where

$$\mathbf{v}(x, z, t) = -\nabla\Phi(x, z, t). \tag{2}$$

The surface of the fluid is described by $z = h + \zeta(x, t)$. It is assumed that the fluid is contained in a structure (lake, river, swimming pool, etc.) with a structureless bottom defined by the $z = 0$ plane and filled to an equilibrium height of $z = h$.

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* Note that these derviations follow Alexander L. Fetter and John Dirk Walecka, **Theoretical Mechanics of Particles and Continua**, (McGraw Hill, 1980), Chapt. 10.

Defining equations for $\Phi(x, z, t)$ and $\zeta(x, t)$

$$0 \leq z \leq h + \zeta(x, t)$$

Continuity equation

$$\nabla \cdot \mathbf{v} = 0 \quad \Rightarrow \quad \frac{\partial^2 \Phi(x, z, t)}{\partial x^2} + \frac{\partial^2 \Phi(x, z, t)}{\partial z^2} = 0 \tag{3}$$

Bernoulli equation (assuming irrotational flow) and gravitation potential energy

$$-\frac{\partial \Phi(x, z, t)}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \Phi(x, z, t)}{\partial x}\right)^2 + \left(\frac{\partial \Phi(x, z, t)}{\partial z}\right)^2\right] + g(z - h) = 0. \tag{4}$$

Boundary conditions on functions

Zero vertical velocity at bottom of tank

$$\frac{\partial \Phi(x,0,t)}{\partial z} = 0. \tag{5}$$

Consistent vertical velocity at water surface

$$v_z(x,z,t)\rfloor_{z=h+\zeta} = \frac{d\zeta}{dt} = \mathbf{v} \cdot \nabla \zeta + \frac{\partial \zeta}{\partial t}. \tag{6}$$

$$-\frac{\partial \Phi(x,z,t)}{\partial z} + \frac{\partial \Phi(x,z,t)}{\partial x}\frac{\partial \zeta(x,t)}{\partial x} - \frac{\partial \zeta(x,t)}{\partial t}\Bigg\rfloor_{z=h+\zeta} = 0 \tag{7}$$

Analysis assuming water height z is small relative to variations in the direction of wave motion (x)

Taylor's expansion about $z = 0$

$$\Phi(x,z,t) \approx \Phi(x,0,t) + z\frac{\partial \Phi}{\partial z}(x,0,t) + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \frac{z^3}{3!}\frac{\partial^3 \Phi}{\partial z^3}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial z^4}(x,0,t)\cdots \quad (8)$$

Note that the zero vertical velocity at the bottom ensures that all odd derivatives $\frac{\partial^n \Phi}{\partial z^n}(x,0,t)$ vanish from the Taylor expansion . In addition, the Laplace equation allows us to convert all even derivatives with respect to z to derivatives with respect to x.

Modified Taylor's expansion

$$\Phi(x,z,t) \approx \Phi(x,0,t) - \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial x^2}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial x^4}(x,0,t)\cdots \quad (9)$$

Linearized equations and their solution

Bernoulli equation evaluated at $z = h + \zeta(x,t)$

$$-\frac{\partial \Phi(x,h,t)}{\partial t} + g\zeta(x,t) = 0 \tag{10}$$

Consistent vertical velocity at $z = h + \zeta(x,t)$

$$-\frac{\partial \Phi(x,z,t)}{\partial z} - \frac{\partial \zeta(x,t)}{\partial t}\bigg\rfloor_{z=h+\zeta} = 0 \tag{11}$$

Using Taylor's expansion results to lowest order

$$-\frac{\partial \Phi(x,z,t)}{\partial z} \approx h\frac{\partial^2 \Phi(x,0,t)}{\partial x^2} \qquad -\frac{\partial \Phi(x,h,t)}{\partial t} \approx -\frac{\partial \Phi(x,0,t)}{\partial t} \tag{12}$$

Decoupled equations

$$\frac{\partial^2 \Phi(x,0,t)}{\partial t^2} = gh\frac{\partial^2 \Phi(x,0,t)}{\partial x^2}. \tag{13}$$

Nonlinear equations and solutions – let $\phi(x,t) \equiv \Phi(x,0,t)$

Approximate forms of Bernoulli equation evaluated at $z = h + \zeta$ surface

$$-\frac{\partial \phi}{\partial t} + \frac{(h+\zeta)^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left((h+\zeta)\frac{\partial^2 \phi}{\partial x^2}\right)^2\right] + g\zeta = 0 \qquad (14)$$

$$-\frac{\partial \phi}{\partial t} + \frac{h^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + g\zeta = 0. \qquad (15)$$

Approximate form of surface velocity expression

$$\frac{\partial}{\partial x}\left((h+\zeta(x,t))\frac{\partial \phi}{\partial x}\right) - \frac{h^3}{3!}\frac{\partial^4 \phi}{\partial x^4} - \frac{\partial \zeta}{\partial t} = 0. \qquad (16)$$

The expressions keep the lowest order nonlinear terms and include up to 4th order derivatives in the linear terms.

Traveling wave solutions with $u = x - ct$: $\phi(x,t) = \chi(u)$ and $\zeta(x,t) = \eta(u)$

Note that the wave "speed" c will be consistently determined

Modified Bernoulli equation

$$c\frac{d\chi(u)}{du} - \frac{ch^2}{2}\frac{d^3\chi(u)}{du^3} + \frac{1}{2}\left(\frac{d\chi(u)}{du}\right)^2 + g\eta(u) = 0. \tag{17}$$

Modified surface velocity equation

$$\frac{d}{du}\left((h+\eta(u))\frac{d\chi(u)}{du}\right) - \frac{h^3}{6}\frac{d^4\chi(u)}{du^4} + c\frac{d\eta(u)}{du} = 0. \tag{18}$$

Integrating and rearranging coupled equations

Modified surface velocity equation

$$\frac{d}{du}\left((h+\eta(u))\frac{d\chi(u)}{du}\right) - \frac{h^3}{6}\frac{d^4\chi(u)}{du^4} + c\frac{d\eta(u)}{du} = 0. \tag{19}$$

$$\Rightarrow (h+\eta)\chi' - \frac{h^3}{6}\chi''' + c\eta = 0 \tag{20}$$

Modified Bernoulli equation

$$\chi' - \frac{h^2}{2}\chi''' + \frac{1}{2c}(\chi')^2 + \frac{g}{c}\eta = 0. \tag{21}$$

$$\Rightarrow \chi' = -\frac{g}{c}\eta + \frac{h^2}{2}\chi''' - \frac{1}{2c}(\chi')^2 \approx -\frac{g}{c}\eta - \frac{h^2 g}{2c}\eta'' - \frac{g^2}{2c^3}\eta^2. \tag{22}$$

Integrating and rearranging coupled equations – continued

Modified surface velocity equation in terms of η

$$(h+\eta)\left(-\frac{g}{c}\eta - \frac{h^2 g}{2c}\eta'' - \frac{g^2}{2c^3}\eta^2\right) + \frac{h^3 g}{6c}\eta'' + c\eta = 0 \tag{23}$$

$$\Rightarrow \left(1 - \frac{gh}{c^2}\right)\eta - \frac{gh^3}{3c^2}\eta'' - \frac{g}{c^2}\left(1 + \frac{gh}{2c^2}\right)\eta^2 = 0. \tag{24}$$

$$\Rightarrow \left(1 - \frac{hg}{c^2}\right)\eta(u) - \frac{h^2}{3}\eta''(u) - \frac{3}{2h}\left[\eta(u)\right]^2 = 0. \tag{25}$$

Solution of the famous Korteweg-de Vries equation

Modified surface velocity equation in terms of η

$$\Rightarrow \left(1 - \frac{hg}{c^2}\right)\eta(u) - \frac{h^2}{3}\eta''(u) - \frac{3}{2h}\left[\eta(u)\right]^2 = 0. \tag{26}$$

Soliton solution

$$\zeta(x,t) = \eta(x-ct) = \eta_0 \,\mathrm{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\,\frac{x-ct}{2h}\right) \tag{27}$$

$$c = \sqrt{\frac{gh}{1-\eta_0/h}} \approx \sqrt{gh}\left(1+\frac{\eta_0}{2h}\right). \tag{28}$$

Here η_0 is a constant to be determined.

Relationship with "standard" form of Korteweg-de Vries equation

New variables

$$\beta = 2\eta_0, \qquad \bar{x} = \sqrt{\frac{3}{2h}}\frac{x}{h}, \qquad \text{and} \qquad \bar{t} = \sqrt{\frac{3}{2h}}\frac{ct}{2\eta_0 h}. \tag{29}$$

Standard Korteweg-de Vries equation

$$\frac{\partial \eta}{\partial \bar{t}} + 6\eta\frac{\partial \eta}{\partial \bar{x}} + \frac{\partial^3 \eta}{\partial \bar{x}^3} = 0. \tag{30}$$

Soliton solution

$$\eta(\bar{x}, \bar{t}) = \frac{\beta}{2}\,\text{sech}^2\left[\frac{\sqrt{\beta}}{2}(\bar{x} - \beta\bar{t})\right]. \tag{31}$$

More details

Modified surface velocity equation in terms of η

$$\left(1 - \frac{hg}{c^2}\right)\eta(u) - \frac{h^2}{3}\eta''(u) - \frac{3}{2h}\left[\eta(u)\right]^2 = 0. \tag{32}$$

Some identities

$$\frac{\eta_0}{h} = 1 - \frac{gh}{c^2}; \quad \frac{\partial\eta}{\partial t} = -c\frac{d\eta}{du}; \quad \frac{\partial\eta}{\partial x} = \frac{d\eta}{du}. \tag{33}$$

Derivative of surface velocity equation

$$\frac{\eta_0}{h}\eta' - \frac{h^2}{3}\eta''' - \frac{3}{h}\eta\eta' = 0. \tag{34}$$

In terms of x and t

$$-\frac{\eta_0}{ch}\frac{\partial\eta}{\partial t} - \frac{h^2}{3}\frac{\partial^3\eta}{\partial x^3} - \frac{3}{h}\eta\frac{\partial\eta}{\partial x} = 0. \tag{35}$$

In terms of $\bar{x}$ and $\bar{t}$

$$\frac{\partial\eta}{\partial\bar{t}} + 6\eta\frac{\partial\eta}{\partial\bar{x}} + \frac{\partial^3\eta}{\partial\bar{x}^3} = 0. \tag{36}$$