

**PHY 711 Classical Mechanics and  
Mathematical Methods  
10-10:50 AM MWF Olin 103**

**Plan for Lecture 9:**

**Continue reading Chapter 3**

- 1. Hamilton's principle**
- 2. Lagrange's equations in presence of magnetic fields**

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**PHY 711 Classical Mechanics and Mathematical Methods**

**MWF 10 AM-10:50 PM OPL 103** <http://www.wfu.edu/~natalie/f12phy711/>

**Instructor:** Natalie Holzworth **Phone:** 758-5510 **Office:** 300 OPL **e-mail:** natalie@wfu.edu

**Course schedule**

(Preliminary schedule -- subject to frequent adjustment.)

Date	F&W Reading	Topic	Assignment
1 Wed, 8/29/2012	Chap. 1	Review of basic principles, Scattering theory	#1
2 Fri, 8/31/2012	Chap. 1	Scattering theory continued	#2
3 Mon, 9/03/2012	Chap. 1	Scattering theory continued	#3
4 Wed, 9/05/2012	Chap. 1 & 2	Scattering theory/Accelerated coordinate frame	#4
5 Fri, 9/07/2012	Chap. 2	Accelerated coordinate frame	#5
6 Mon, 9/10/2012	Chap. 3	Calculus of Variation	#6
7 Wed, 9/12/2012	Chap. 3	Calculus of Variation continued	#7
8 Fri, 9/14/2012	Chap. 3	Lagrangian	#8
9 Mon, 9/17/2012	Chap. 3	Lagrangian	#8

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**Hamilton's principle:**

Given the Lagrangian function :  $L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) \equiv T - U$ ,  
The physical trajectories of the generalized coordinates  $\{q_\sigma(t)\}$

Are those which minimize the action :  $S = \int_{\sigma} L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) dt$

Euler - Lagrange equations :

$$\sum_{\sigma} \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma = 0 \Rightarrow \text{for each } \sigma : \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0$$

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**Example – simple harmonic oscillator**  
 $T = \frac{1}{2} m \dot{x}^2$        $U = \frac{1}{2} m \omega^2 x^2$

Assume  $x(0) = 0$  and  $x(\frac{\pi}{\omega}) = 0$        $S = \int_0^{\pi/\omega} (T - U) dt$

Trial functions     $x_1(t) = A \sin(\omega t)$        $S_1 = 0$   
                        $x_2(t) = At \cdot (\frac{\pi}{\omega} - t)$        $S_2 = 0.002 A^2 m$   
                        $x_3(t) = Ae^{-\alpha t} \sin(\omega t)$        $S_3 = 0.196 A^2 m$

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Note: in “proof” of Hamilton’s principle:

$$\left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0 \quad \text{for } L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) \equiv T - U$$

It was necessary to assume that:

$\frac{d}{dt} \frac{\partial U}{\partial \dot{q}_\sigma}$  did not contribute to the result.

⇒ How can we represent velocity-dependent forces?

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**Lorentz forces:**  
 For particle of charge  $q$  in an electric field  $\mathbf{E}(\mathbf{r}, t)$  and magnetic field  $\mathbf{B}(\mathbf{r}, t)$ :

Lorentz force:  $\mathbf{F} = q(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B})$   
 x – component:  $F_x = q(E_x + \frac{1}{c} (\mathbf{v} \times \mathbf{B})_x)$

In this case, it is convenient to use cartesian coordinates  
 $L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$   
 $T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

x - component:  $\left( \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \right) = 0$

Apparently:  $F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}}$

Answer:  $U = q\Phi(\mathbf{r}, t) - \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$

where  $\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}$        $\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$

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Lorentz forces, continued:

$$x\text{-component of Lorentz force: } F_x = q(E_x + \frac{1}{c}(\mathbf{v} \times \mathbf{B})_x)$$

$$\text{Suppose: } U = q\Phi(\mathbf{r}, t) - \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

$$\text{Consider: } F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}}$$

$$-\frac{\partial U}{\partial x} = -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} + \frac{q}{c} \left( \dot{x} \frac{\partial A_x(\mathbf{r}, t)}{\partial x} + \dot{y} \frac{\partial A_y(\mathbf{r}, t)}{\partial x} + \dot{z} \frac{\partial A_z(\mathbf{r}, t)}{\partial x} \right)$$

$$\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c} A_x(\mathbf{r}, t)$$

$$\frac{d}{dt} \frac{\partial U}{\partial \dot{x}} = \frac{q}{c} \frac{dA_x(\mathbf{r}, t)}{dt} = \frac{q}{c} \left( \frac{\partial A_x(\mathbf{r}, t)}{\partial x} \dot{x} + \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \dot{y} + \frac{\partial A_x(\mathbf{r}, t)}{\partial z} \dot{z} + \frac{\partial A_x(\mathbf{r}, t)}{\partial t} \right)$$

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Lorentz forces, continued:

$$-\frac{\partial U}{\partial x} = -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} + \frac{q}{c} \left( \dot{x} \frac{\partial A_x(\mathbf{r}, t)}{\partial x} + \dot{y} \frac{\partial A_y(\mathbf{r}, t)}{\partial x} + \dot{z} \frac{\partial A_z(\mathbf{r}, t)}{\partial x} \right)$$

$$\frac{d}{dt} \frac{\partial U}{\partial \dot{x}} = -\frac{q}{c} \left( \frac{\partial A_x(\mathbf{r}, t)}{\partial x} \dot{x} + \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \dot{y} + \frac{\partial A_x(\mathbf{r}, t)}{\partial z} \dot{z} + \frac{\partial A_x(\mathbf{r}, t)}{\partial t} \right)$$

$$\begin{aligned} F_x &= -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}} \\ &= -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} + \frac{q}{c} \dot{y} \left( \frac{\partial A_y(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \right) + \frac{q}{c} \dot{z} \left( \frac{\partial A_z(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial z} \right) - \frac{q}{c} \frac{\partial A_x(\mathbf{r}, t)}{\partial t} \\ &= -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} - \frac{q}{c} \frac{\partial A_x(\mathbf{r}, t)}{\partial t} + \frac{q}{c} \dot{y} \left( \frac{\partial A_y(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \right) + \frac{q}{c} \dot{z} \left( \frac{\partial A_z(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial z} \right) \\ &= qE_x(\mathbf{r}, t) + \frac{q}{c} (\dot{y}B_z(\mathbf{r}, t) - \dot{z}B_y(\mathbf{r}, t)) = qE_x(\mathbf{r}, t) + \frac{q}{c} (\mathbf{v} \times \mathbf{B}(\mathbf{r}, t))_x \end{aligned}$$

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Lorentz forces, continued:

In this case, it is convenient to use cartesian coordinates

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad U = q\Phi(\mathbf{r}, t) - \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

$$\text{where } \mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\Phi(\mathbf{r}, t) + \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

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Example Lorentz force

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\Phi(\mathbf{r}, t) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

Suppose  $\mathbf{E}(\mathbf{r}, t) \equiv 0$ ,  $\mathbf{B}(\mathbf{r}, t) \equiv B_0\hat{\mathbf{z}}$

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{2}B_0(y\hat{\mathbf{x}} - x\hat{\mathbf{y}})$$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c}B_0(\dot{x}y - y\dot{x})$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(m\dot{x} + \frac{q}{2c}B_0y\right) + \frac{q}{2c}B_0\dot{y} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(m\dot{y} - \frac{q}{2c}B_0x\right) - \frac{q}{2c}B_0\dot{x} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \quad \Rightarrow \quad \frac{d}{dt}m\dot{z} = 0$$

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Example Lorentz force -- continued

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c}B_0(\dot{x}y - y\dot{x})$$

$$m\ddot{x} = -\frac{q}{c}B_0\dot{y}$$

$$m\ddot{y} = \frac{q}{c}B_0\dot{x}$$

$$m\ddot{z} = 0$$

Check :

Same result for

Direct application of Newton's laws:  $m\ddot{\mathbf{r}} = \frac{q}{c}\dot{\mathbf{r}} \times B_0\hat{\mathbf{z}}$

Different Gauge:  $\mathbf{A}(\mathbf{r}) = B_0y\hat{\mathbf{x}}$

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