PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 9:

Continue reading Chapter 3

1. Hamilton's principle

2. Lagrange's equations in presence of magnetic fields

PHY 711 Classical Mechanics and Mathematical Methods

| MWF 10 AM-10:50 PM | OPL 103 | http://www.wfu.edu/~natalie/f12phy711/ |

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/29/2012	Chap. 1	Review of basic principles;Scattering theory	#1
2	Fri, 8/31/2012	Chap. 1	Scattering theory continued	#2
3	Mon, 9/03/2012	Chap. 1	Scattering theory continued	#3
4	Wed, 9/05/2012	Chap. 1 & 2	Scattering theory/Accelerated coordinate frame	#4
5	Fri, 9/07/2012	Chap. 2	Accelerated coordinate frame	#5
6	Mon, 9/10/2012	Chap. 3	Calculus of Variation	#6
7	Wed, 9/12/2012	Chap. 3	Calculus of Variation continued	
8	Fri, 9/14/2012	Chap. 3	Lagrangian	#7
9	Mon, 9/17/2012	Chap. 3	Lagrangian	#8

Hamilton's principle:

Given the Lagrangian function : $\quad L = L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) \equiv T - U$,

The physical trajectories of the generalized coordinates $\{q_\sigma(t)\}$

Are those which minimize the action : $S = \iiint_\sigma L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)dt$

Euler - Lagrange equations :

$$\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma}\right)\delta q_\sigma = 0 \quad \Rightarrow \text{for each } \sigma : \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma}\right) = 0$$

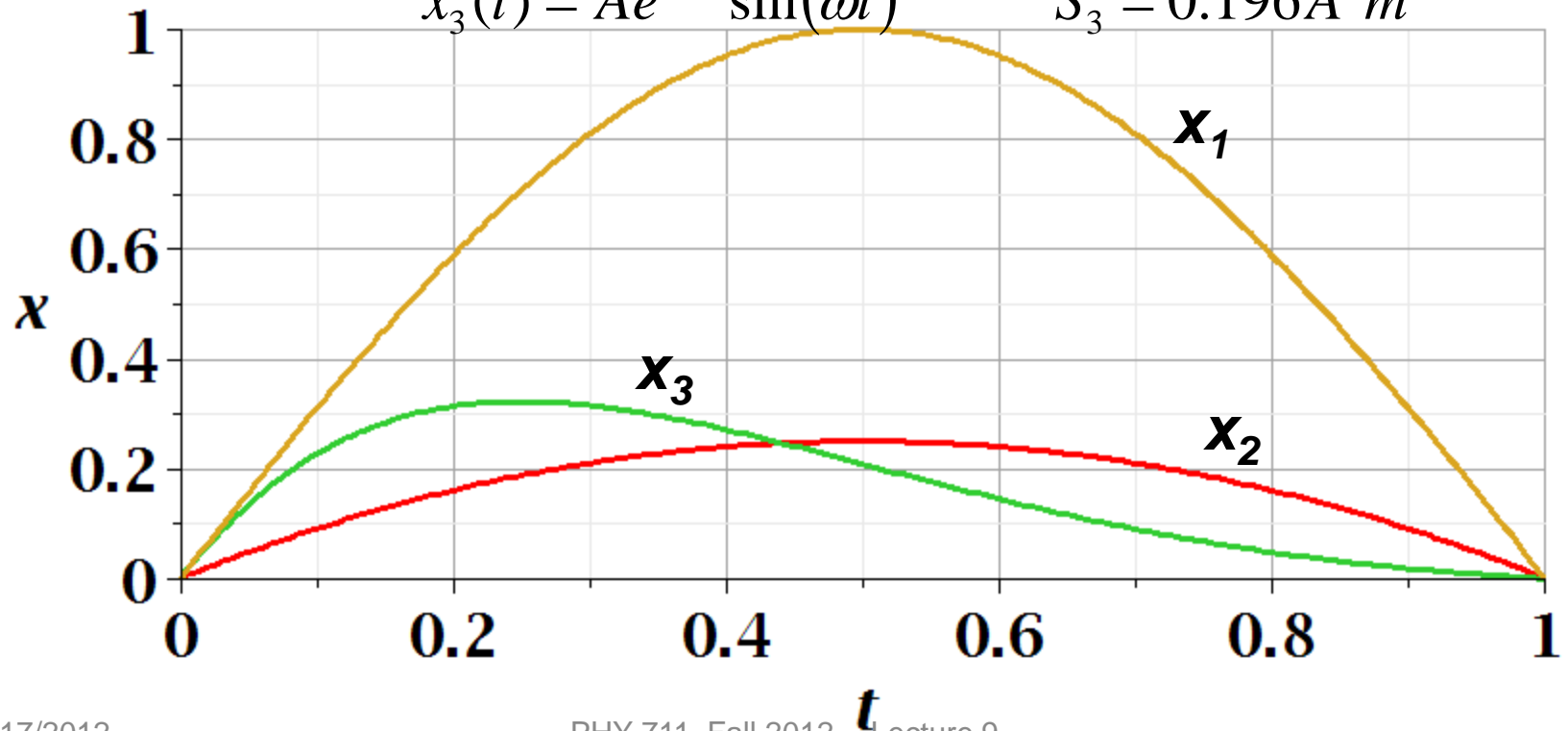# Example – simple harmonic oscillator

$$T = \tfrac{1}{2}m\dot{x}^2 \qquad\qquad U = \tfrac{1}{2}m\omega^2 x^2$$

Assume $x(0) = 0$ and $x(\tfrac{\pi}{\omega}) = 0$ $\qquad S = \displaystyle\int_0^{\pi/\omega}(T - U)\,dt$

Trial functions $\quad x_1(t) = A\sin(\omega t) \qquad\qquad S_1 = 0$

$$x_2(t) = At\cdot\left(\tfrac{\pi}{\omega} - t\right) \qquad\qquad S_2 = 0.002A^2 m$$

$$x_3(t) = Ae^{-\omega t}\sin(\omega t) \qquad\qquad S_3 = 0.196A^2 m$$

Note: in "proof" of Hamilton's principle:

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0 \quad \text{for} \quad L = L\left(\{q_\sigma\}, \{\dot{q}_\sigma\}, t\right) \equiv T - U$$

It was necessary to assume that :

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{q}_\sigma} \quad \text{did not contribute to the result.}$$

$$\Rightarrow \text{How can we represent velocity - dependent forces?}$$

Lorentz forces:

For particle of charge q in an electric field $\mathbf{E}(\mathbf{r}, t)$ and magnetic field $\mathbf{B}(\mathbf{r}, t)$:

Lorentz force: $\quad \mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)$

x – component: $\quad F_x = q\left(E_x + \frac{1}{c}(\mathbf{v} \times \mathbf{B})_x\right)$

In this case, it is convenient to use cartesian coordinates

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

x - component: $\quad \left(\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}} - \dfrac{\partial L}{\partial x}\right) = 0$

Apparently: $\quad F_x = -\dfrac{\partial U}{\partial x} + \dfrac{d}{dt}\dfrac{\partial U}{\partial \dot{x}}$

Answer: $\quad U = q\Phi(\mathbf{r}, t) - \dfrac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$

where $\quad \mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \dfrac{1}{c}\dfrac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \qquad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$

Lorentz forces, continued:

$x-$ component of Lorentz force : $F_x = q\left(E_x + \tfrac{1}{c}\left(\mathbf{v}\times\mathbf{B}\right)_x\right)$

Suppose : $\quad U = q\Phi(\mathbf{r},t) - \dfrac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$

Consider : $\quad F_x = -\dfrac{\partial U}{\partial x} + \dfrac{d}{dt}\dfrac{\partial U}{\partial \dot{x}}$

$$-\frac{\partial U}{\partial x} = -q\frac{\partial\Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\left(\dot{x}\frac{\partial A_x(\mathbf{r},t)}{\partial x} + \dot{y}\frac{\partial A_y(\mathbf{r},t)}{\partial x} + \dot{z}\frac{\partial A_z(\mathbf{r},t)}{\partial x}\right)$$

$$\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}A_x(\mathbf{r},t)$$

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}\frac{dA_x(\mathbf{r},t)}{dt} = -\frac{q}{c}\left(\frac{\partial A_x(\mathbf{r},t)}{\partial x}\dot{x} + \frac{\partial A_x(\mathbf{r},t)}{\partial y}\dot{y} + \frac{\partial A_x(\mathbf{r},t)}{\partial z}\dot{z} + \frac{\partial A_x(\mathbf{r},t)}{\partial t}\right)$$

Lorentz forces, continued:

$$-\frac{\partial U}{\partial x} = -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\left(\dot{x}\frac{\partial A_x(\mathbf{r},t)}{\partial x} + \dot{y}\frac{\partial A_y(\mathbf{r},t)}{\partial x} + \dot{z}\frac{\partial A_z(\mathbf{r},t)}{\partial x}\right)$$

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}\left(\frac{\partial A_x(\mathbf{r},t)}{\partial x}\dot{x} + \frac{\partial A_x(\mathbf{r},t)}{\partial y}\dot{y} + \frac{\partial A_x(\mathbf{r},t)}{\partial z}\dot{z} + \frac{\partial A_x(\mathbf{r},t)}{\partial t}\right)$$

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t} + \frac{q}{c}\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial z}\right)$$

$$= qE_x(\mathbf{r},t) + \frac{q}{c}\left(\dot{y}B_z(\mathbf{r},t) - \dot{z}B_y(\mathbf{r},t)\right) = qE_x(\mathbf{r},t) + \frac{q}{c}\left(\mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right)_x$$

Lorentz forces, continued:

In this case, it is convenient to use cartesian coordinates

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \qquad U = q\Phi(\mathbf{r}, t) - \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

where $\quad \mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \dfrac{1}{c}\dfrac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \qquad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$

$$L = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\Phi(\mathbf{r}, t) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

Example Lorentz force

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi(\mathbf{r},t) + \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

$$\text{Suppose} \quad \mathbf{E}(\mathbf{r},t) \equiv 0, \quad \mathbf{B}(\mathbf{r},t) \equiv B_0\hat{\mathbf{z}}$$

$$\mathbf{A}(\mathbf{r},t) = \tfrac{1}{2} B_0\left(y\hat{\mathbf{x}} - x\hat{\mathbf{y}}\right)$$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(\dot{x}y - \dot{y}x\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{x} + \frac{q}{2c}B_0 y\right) + \frac{q}{2c}B_0\dot{y} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{y} - \frac{q}{2c}B_0 x\right) - \frac{q}{2c}B_0\dot{x} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \qquad \Rightarrow \frac{d}{dt}m\dot{z} = 0$$

Example Lorentz force -- continued

$$L = \tfrac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) + \frac{q}{2c} B_0 \left(\dot{x} y - \dot{y} x \right)$$

$$m\ddot{x} = -\frac{q}{c} B_0 \dot{y}$$

$$m\ddot{y} = \frac{q}{c} B_0 \dot{x}$$

$$m\ddot{z} = 0$$

Check :

Same result for

Direct application of Newton's laws : $m\ddot{\mathbf{r}} = \dfrac{q}{c} \dot{\mathbf{r}} \times B_0 \hat{\mathbf{z}}$

Different Gauge : $\mathbf{A}(\mathbf{r}) = B_0 y \hat{\mathbf{x}}$