PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 8:

Continue reading Chapter 3

1. **Lagrange's equations**
2. **D'Alembert's principle**

PHY 711 Classical Mechanics and Mathematical Methods

MWF 10 AM-10:50 PM	OPL 103	http://www.wfu.edu/~natalie/f12phy711/

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/29/2012	Chap. 1	Review of basic principles;Scattering theory	#1
2	Fri, 8/31/2012	Chap. 1	Scattering theory continued	#2
3	Mon, 9/03/2012	Chap. 1	Scattering theory continued	#3
4	Wed, 9/05/2012	Chap. 1 & 2	Scattering theory/Accelerated coordinate frame	#4
5	Fri, 9/07/2012	Chap. 2	Accelerated coordinate frame	#5
6	Mon, 9/10/2012	Chap. 3	Calculus of Variation	#6
7	Wed, 9/12/2012	Chap. 3	Calculus of Variation continued	
8	Fri, 9/14/2012	Chap. 3	Lagrangian	#7

Summary of results from the calculus of variation

For the class of problems where we need to perform an extremization on an integral form:

$$I = \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx$$

A necessary condition is the Euler-Lagrange equations:

$$\left(\frac{\partial f}{\partial y}\right) - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)\right] = 0$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = \left(\frac{\partial f}{\partial x}\right)$$

Application to particle dynamics

Simple example: vertical trajectory of particle of mass *m* subject to constant downward acceleration *a*=-*g*.

$$m\frac{d^2y}{dt^2} = -mg$$

$$y(t) = y_i + v_i t - \tfrac{1}{2} g t^2$$

Now consider the Lagrangian defined to be:

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U$$

Kinetic energy

Potential energy

In our example:

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy$$

Hamilton's principle states:

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy\right)dt \quad \text{is minimized for physical } y(t):$$

Condition for minimizing the action :

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2} m \left(\frac{dy}{dt} \right)^2 - mgy \right) dt$$

Euler - Lagrange relations :

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0$$

$$\Rightarrow -mg - \frac{d}{dt} m\dot{y} = 0$$

$$\Rightarrow \frac{d}{dt}\frac{dy}{dt} = -g$$

$$y(t) = y_i + v_i t - \tfrac{1}{2} g t^2$$

Check :

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2} m \left(\frac{dy}{dt} \right)^2 - mgy \right) dt$$

Assume $t_i = 0, \;\; y_i = H \equiv \tfrac{1}{2} g T^2; \;\; t_f = T, \;\; y_f = 0$

Trial trajectories : $y_1(t) = \tfrac{1}{2} g T^2 (1 - t/T)$

$$y_2(t) = \tfrac{1}{2} g T^2 (1 - t^2/T^2)$$

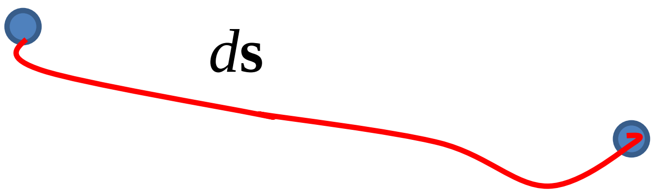
$$y_3(t) = \tfrac{1}{2} g T^2 (1 - t^3/T^3)$$

Maple says :

$$S_1 = -0.125 g^2 T^3$$

$$S_2 = -0.167 g^2 T^3$$

$$S_3 = -0.150 g^2 T^3$$



Generalized coordinates:

$$q_\sigma(\{x_i\})$$

Newton's laws:

$$\mathbf{F}\text{-}m\mathbf{a} = 0 \qquad \Rightarrow (\mathbf{F}\text{-}m\mathbf{a})\cdot d\mathbf{s} = 0$$

$$\mathbf{F}\cdot d\mathbf{s} = \sum_\sigma \sum_i F_i \frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma$$

For a conservative force: $F_i = -\dfrac{\partial U}{\partial x_i}$

$$\mathbf{F}\cdot d\mathbf{s} = -\sum_\sigma \sum_i \frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma$$

Generalized coordinates:

$$q_\sigma(\{x_i\})$$

Newton's laws:

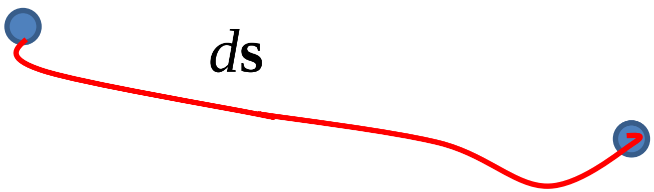
$$\mathbf{F}\text{-}m\mathbf{a} = 0 \qquad \Rightarrow (\mathbf{F}\text{-}m\mathbf{a})\cdot d\mathbf{s} = 0$$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \sum_i m\ddot{x}_i \frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma$$

$$= \sum_\sigma \sum_i \left(\frac{d}{dt}\left(m\dot{x}_i \frac{\partial x_i}{\partial q_\sigma} \right) - m\dot{x}_i \frac{d}{dt}\frac{\partial x_i}{\partial q_\sigma} \right)\delta q_\sigma$$

Claim: $\dfrac{\partial x_i}{\partial q_\sigma} = \dfrac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$ and $\dfrac{d}{dt}\dfrac{\partial x_i}{\partial q_\sigma} = \dfrac{\partial}{\partial q_\sigma}\dfrac{dx_i}{dt} \equiv \dfrac{\partial \dot{x}_i}{\partial q_\sigma}$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \sum_i \left(\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2} m\dot{x}_i^{\,2}\right)}{\partial \dot{q}_\sigma} \right) - \frac{\partial\left(\frac{1}{2} m\dot{x}_i^{\,2}\right)}{\partial q_\sigma} \right)\delta q_\sigma$$



Generalized coordinates:

$q_\sigma(\{x_i\})$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \sum_i \left(\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}m\dot{x}_i{}^2\right)}{\partial \dot{q}_\sigma} \right) - \frac{\partial\left(\frac{1}{2}m\dot{x}_i{}^2\right)}{\partial q_\sigma} \right)\delta q_\sigma$$

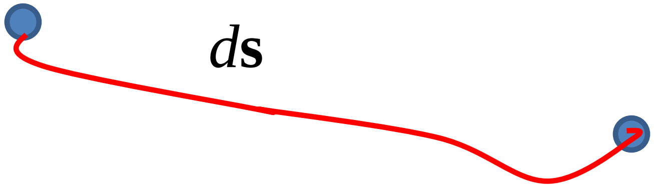
Define -- kinetic energy: $\quad T \equiv \sum_i \frac{1}{2}m\dot{x}_i{}^2$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right)\delta q_\sigma$$

Recall:

$$\mathbf{F}\cdot d\mathbf{s} = \sum_\sigma \sum_i \frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma = \sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma$$

$$(\mathbf{F}\text{-}m\mathbf{a})\cdot d\mathbf{s} = \sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma - \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right)\delta q_\sigma = 0$$



Generalized coordinates :

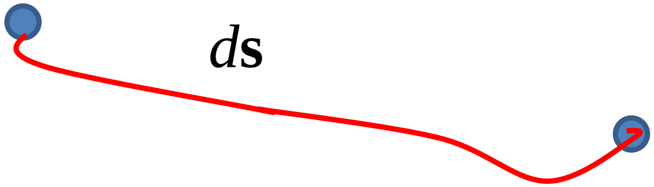
$q_\sigma(\{x_i\})$

$$\left(\mathbf{F}\text{-}m\mathbf{a}\right)\cdot d\mathbf{s} = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma - \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma}\right)\delta q_\sigma = 0$$

$$= -\sum_\sigma \left(\frac{d}{dt}\frac{\partial (T-U)}{\partial \dot{q}_\sigma} - \frac{\partial (T-U)}{\partial q_\sigma}\right)\delta q_\sigma = 0$$

Note : This is only true if

$$\frac{\partial U}{\partial \dot{q}_\sigma} = 0$$



Generalized coordinates :

$$q_\sigma\left(\{x_i\}\right)$$

Define -- Lagrangian : $L \equiv T - U$

$$L = L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)$$

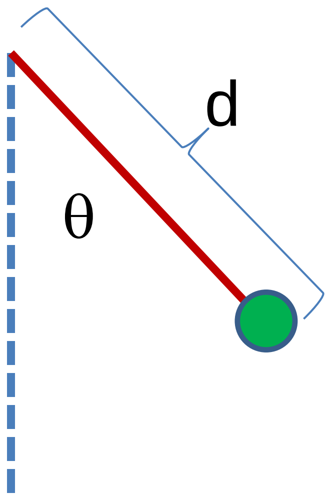
$$\left(\mathbf{F}\text{-}m\mathbf{a}\right)\cdot d\mathbf{s} = -\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma}\right)\delta q_\sigma = 0$$

$\Rightarrow$ Minimization integral : $S = \int_{t_i}^{t_f} L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)dt$

Euler – Lagrange equations : $L = L(\{q_\sigma\},\{\dot{q}_\sigma\},t) = T - U$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Example:



$$L = L(\theta,\dot{\theta}) = \tfrac{1}{2} m d^2 \dot{\theta}^2 - mg(d - d\cos\theta)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \quad \Rightarrow \frac{d}{dt} m d^2 \dot{\theta} - mgd\sin\theta = 0$$

$$\frac{d^2\theta}{dt^2} = \frac{g}{d}\sin\theta$$

Another example: $L = L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) = T - U$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$$L = L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \tfrac{1}{2} I_1\left(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2\right) + \tfrac{1}{2} I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2 - Mgd\cos\beta$$