

**PHY 711 Classical Mechanics and  
Mathematical Methods  
10-10:50 AM MWF Olin 103**

**Plan for Lecture 6:**

**Start reading Chapter 3 –  
First focusing on the “calculus of  
variation”**

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**Course schedule**

(Preliminary schedule -- subject to frequent adjustment.)

| Date               | F&W Reading | Topic  | Assignment |
|--------------------|-------------|--|------------|
| 1   Wed, 8/29/2012 | Chap. 1     | Review of basic principles; Scattering theory  | #1         |
| 2   Fri, 8/31/2012 | Chap. 1     | Scattering theory continued                    | #2         |
| 3   Mon, 9/03/2012 | Chap. 1     | Scattering theory continued                    | #3         |
| 4   Wed, 9/05/2012 | Chap. 1 & 2 | Scattering theory/Accelerated coordinate frame | #4         |
| 5   Fri, 9/07/2012 | Chap. 2     | Accelerated coordinate frame                   | #5         |
| 6   Mon, 9/10/2012 | Chap. 3     | Calculus of Variation                          | #6         |
| 7   Wed, 9/12/2012 | Chap. 3     | Calculus of Variation continued                |            |

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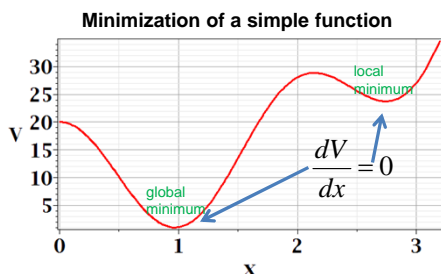
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In Chapter 3, the notion of Lagrangian dynamics is developed; reformulating Newton’s laws in terms of minimization of related functions. In preparation, we need to develop a mathematical tool known as “the calculus of variation”.



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**Minimization of a simple function**  
 Given a function  $V(x)$ , find the value(s) of  $x$  for which  $V(x)$  is minimized (or maximized).  
 Necessary condition:  $\frac{dV}{dx} = 0$

The graph shows a function  $V(x)$  on a coordinate system where the x-axis ranges from 0 to 3 and the y-axis ranges from 0 to 30. The function has a global minimum at  $x=1$  and a local minimum at  $x=3$ . At both minima, the derivative  $\frac{dV}{dx}$  is zero, indicated by blue arrows pointing to the x-axis at those points.

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**Functional minimization**  
 Consider a family of functions  $y(x)$ , with the end points  $y(x_i) = y_i$  and  $y(x_f) = y_f$  and a function  $L\left\{y(x), \frac{dy}{dx}, x\right\}$ .  
 Find the function  $y(x)$  which extremizes  $L\left\{y(x), \frac{dy}{dx}, x\right\}$ .  
 Necessary condition:  $\delta L = 0$

Example:  

$$L = \int_{(0,0)}^{1,1} \sqrt{(dx)^2 + (dy)^2} \, y$$

The graph shows three curves connecting the points (0,0) and (1,1) on a coordinate system where both axes range from 0 to 1. The curves are colored yellow, red, and green. The red curve is a straight line, while the yellow and green curves are curved.

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Example:  

$$L = \int_{(0,0)}^{1,1} \sqrt{(dx)^2 + (dy)^2} \, y$$

$$= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

Sample functions:

$y_1(x) = \sqrt{x} \quad L = \int_0^1 \sqrt{1 + \frac{1}{4x}} \, dx = 1.4789$

$y_2(x) = x \quad L = \int_0^1 \sqrt{1 + 1} \, dx = \sqrt{2} = 1.4142$

$y_2(x) = x^2 \quad L = \int_0^1 \sqrt{1 + 4x^2} \, dx = 1.4789$

The graph shows three curves connecting the points (0,0) and (1,1) on a coordinate system where both axes range from 0 to 1. The curves are colored yellow, red, and green. The red curve is a straight line, while the yellow and green curves are curved.

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**Calculus of variation example for a pure integral functions**

Find the function  $y(x)$  which extremizes  $L\left\{y(x), \frac{dy}{dx}, x\right\}$

where  $L\left\{y(x), \frac{dy}{dx}, x\right\} \equiv \int_{x_i}^{x_f} f\left\{y(x), \frac{dy}{dx}, x\right\} dx$ .

Necessary condition :  $\delta L = 0$

At any  $x$ , let  $y(x) \rightarrow y(x) + \delta y(x)$

$$\frac{dy(x)}{dx} \rightarrow \frac{dy(x)}{dx} + \delta \frac{dy(x)}{dx}$$

Formally:

$$\delta L = \int_{x_i}^{x_f} \left[ \left( \frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \delta \left( \frac{dy}{dx} \right) \right] \right] dx.$$

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After some derivations, we find

$$\begin{aligned} \delta L &= \int_{x_i}^{x_f} \left[ \left( \frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \delta \left( \frac{dy}{dx} \right) \right] \right] dx \\ &= \int_{x_i}^{x_f} \left[ \left( \frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \right] \right] \delta y dx = 0 \quad \text{for all } x_i \leq x \leq x_f \\ \Rightarrow \left( \frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \right] &= 0 \quad \text{for all } x_i \leq x \leq x_f \end{aligned}$$

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Example :

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx \Rightarrow f\left\{y(x), \frac{dy}{dx}, x\right\} = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \\ \left( \frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[ \left( \frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \right] &= 0 \\ \Rightarrow - \frac{d}{dx} \left( \frac{dy/dx}{\sqrt{1 + (dy/dx)^2}} \right) &= 0 \end{aligned}$$

Solution :

$$\begin{aligned} \left( \frac{dy/dx}{\sqrt{1 + (dy/dx)^2}} \right) &= K \quad \frac{dy}{dx} = K' \equiv \frac{K}{\sqrt{1 - K^2}} \\ \Rightarrow y(x) &= x \end{aligned}$$

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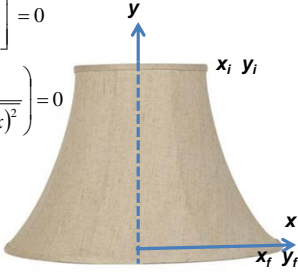
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Example :

$$A = 2\pi \int_{x_i}^{x_f} x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \Rightarrow f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[ \left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y} \right] = 0$$

$$\Rightarrow -\frac{d}{dx} \left( \frac{x dy/dx}{\sqrt{1 + (dy/dx)^2}} \right) = 0$$


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$$-\frac{d}{dx} \left( \frac{x dy/dx}{\sqrt{1 + (dy/dx)^2}} \right) = 0$$

$$\frac{x dy/dx}{\sqrt{1 + (dy/dx)^2}} = K_1$$

$$\frac{dy}{dx} = -\frac{1}{\left(\frac{x}{K_1}\right)^2 - 1}$$

$$\Rightarrow y(x) = K_2 - K_1 \ln\left(x + \sqrt{x^2 - K_1^2}\right)$$

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Review : for  $f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$ ,

a necessary condition to extremize  $\int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx$  :

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[ \left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y} \right] = 0$$

Note that for  $f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$ ,

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right) \frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right) \frac{d}{dx} \frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx} \left(\frac{\partial f}{\partial (dy/dx)}\right)\right) \frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right) \frac{d}{dx} \frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx} \left( f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = \left(\frac{\partial f}{\partial x}\right)$$

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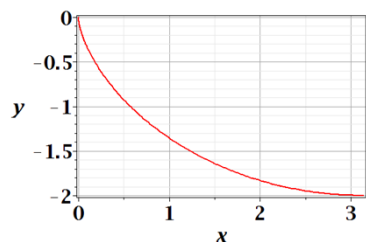
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**Brachistochrone problem:** (solved by Newton in 1696)

<http://mathworld.wolfram.com/BrachistochroneProblem.html>



A particle of weight  $mg$  travels frictionlessly down a path of shape  $y(x)$ . What is the shape of the path  $y(x)$  that minimizes the travel time from  $y(0)=0$  to  $y(\pi)=-2$ ?

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$$T = \int_{x_i, y_i}^{x_f, y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{2gy}} dx \quad \text{because } \frac{1}{2}mv^2 = mgy$$

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{y}}$$

$$\frac{d}{dx} \left( f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = 0$$

$$\frac{d}{dx} \left( \frac{1}{\sqrt{y \left( 1 + \left( \frac{dy}{dx} \right)^2 \right)}} \right) = 0$$

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