PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

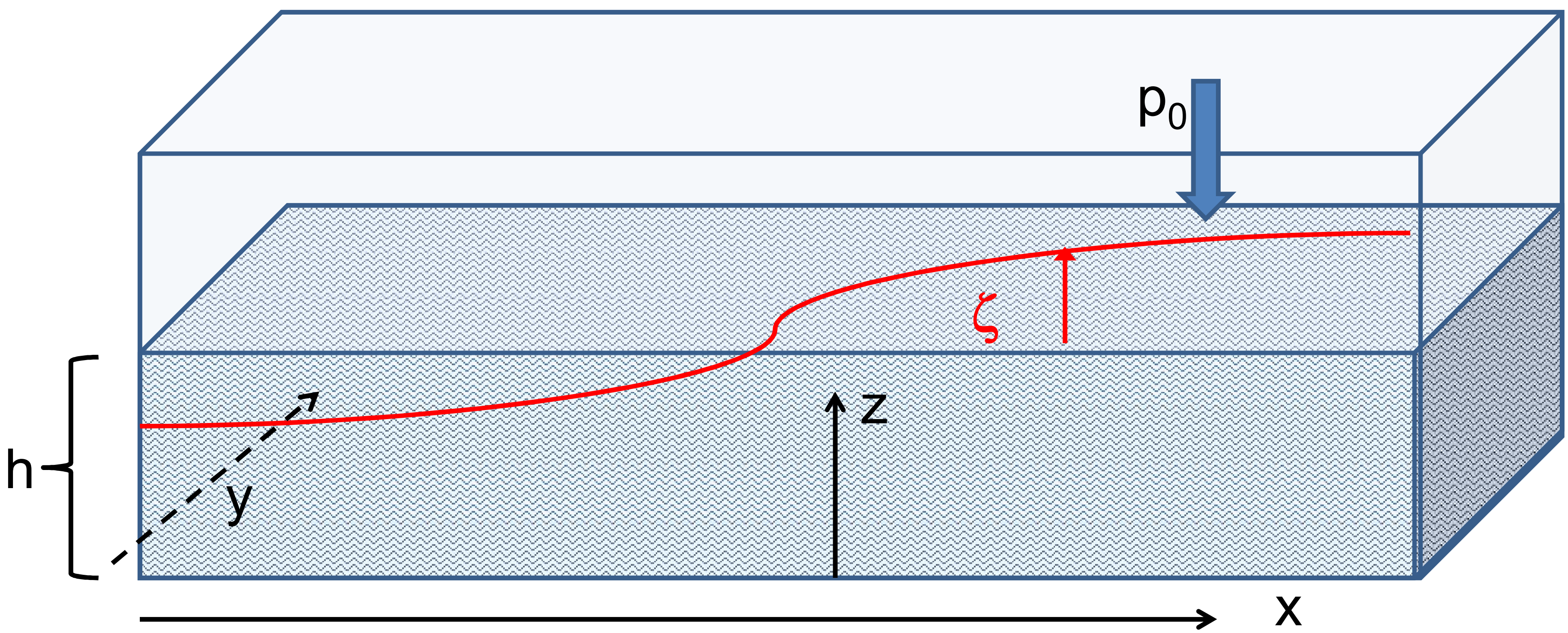
Plan for Lecture 36:

Chapter 10 in F & W: Surface waves

1. **Water waves in a shallow or deep channel – linear equations**
2. **Nonlinear effects – soliton solutions**

22	Wed, 10/17/2012	Chap. 7, 5	Moment of inertia	
	Fri, 10/19/2012		Fall break	
23	Mon, 10/22/2012	Chap. 5	Rigid body rotation	#16
24	Wed, 10/24/2012	Chap. 5	Rigid body rotation	#17
25	Fri, 10/26/2012	Chap. 5	Rigid body rotation	#18
26	Mon, 10/29/2012	Chap. 8	Waves in elastic membranes	#19
27	Wed, 10/31/2012	Chap. 9	Introduction to hydrodynamics	
28	Fri, 11/01/2012	Chap. 9	Introduction to hydrodynamics	
29	Mon, 11/05/2012	Chap. 9	Introduction to hydrodynamics	#20
30	Wed, 11/07/2012	Chap. 9	Sound waves	
31	Fri, 11/09/2012	Chap. 9	Linear sound waves	#21
32	Mon, 11/12/2012	Chap. 9	Green's function for linear sound	
33	Wed, 11/14/2012	Chap. 9	Non-linear sound	
34	Fri, 11/16/2012	Chap. 9	Non-linear sound	Take Home Exam
35	Mon, 11/19/2012	Chap. 10	Surface waves	Take Home Exam
	Wed, 11/21/2012		*Thanksgiving Holiday*	
	Fri, 11/23/2012		*Thanksgiving Holiday*	
36	Mon, 11/26/2012	Chap. 10	Surface waves	Exam due
37	Wed, 11/28/2012	Chap. 10	Surface waves	
38	Fri, 11/30/2012	Chap. 10	Surface waves	
39	Mon, 12/03/2012		Student presentations I	
40	Wed, 12/05/2012		Student presentations II	

Consider a container of water with average height h and surface h+ζ(x,y,t) (slightly different notation than last time):



Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) + \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = -\nabla U - \frac{\nabla p}{\rho}$$

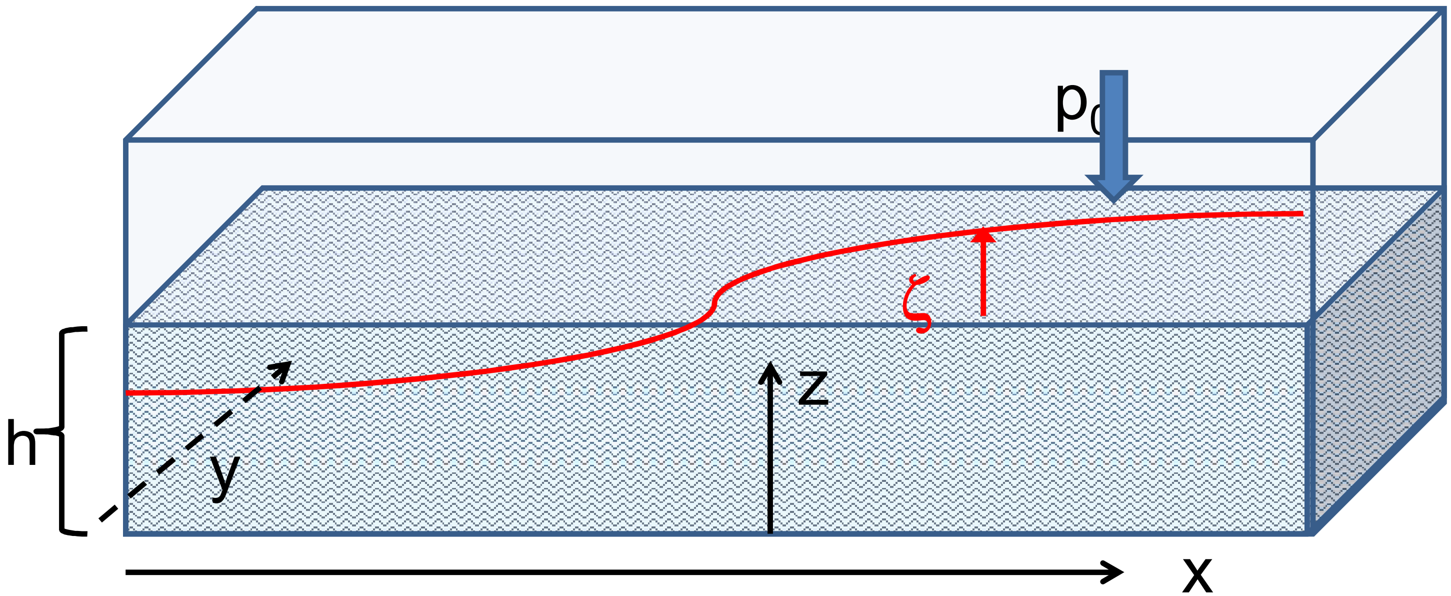
Assume that $\nabla\times\mathbf{v} = 0$ (irrotational flow) $\Rightarrow \mathbf{v} = -\nabla\Phi$

$$\Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + U + \frac{p}{\rho}\right) = 0$$

$$\Rightarrow -\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + U + \frac{p}{\rho} = \text{constant (within the fluid)}$$

For the same system, the continuity condition becomes

$$\nabla\cdot\mathbf{v} = -\nabla^2\Phi = 0$$



Within fluid: $\quad 0 \le z \le h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2} v^2 + g(z-h) = \text{constant} \quad \text{(We have absorbed } p_0 \text{ in our constant.)}$$

$$-\nabla^2 \Phi = 0$$

At surface: $\quad z = h + \zeta \qquad$ with $\zeta = \zeta(x, y, t)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} \qquad \text{where } v_{x,y} = v_{x,y}(x, y, h+\zeta, t)$$

Full equations:

Within fluid: $\quad 0 \le z \le h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2} v^2 + g(z-h) = \text{constant} \quad \text{(We have absorbed}$$

$$-\nabla^2 \Phi = 0 \qquad\qquad p_0 \text{ in our constant.)}$$

At surface: $\quad z = h + \zeta \qquad$ with $\zeta = \zeta(x, y, t)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} \qquad \text{where } v_{x,y} = v_{x,y}(x, y, h+\zeta, t)$$

Linearized equations:

For $\quad 0 \le z \le h + \zeta$: $\quad -\dfrac{\partial \Phi}{\partial t} + g(z-h) = 0 \qquad -\nabla^2 \Phi = 0$

At surface: $\quad z = h + \zeta \qquad \dfrac{d\zeta}{dt} = \dfrac{\partial \zeta}{\partial t} = v_z(x, y, h+\zeta, t)$

$$-\frac{\partial \Phi(x, y, h+\zeta, t)}{\partial t} + g\zeta = 0$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x*:

$$\text{For}\quad 0 \le z \le h + \zeta: \qquad \nabla^2\Phi = \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2}\right)\Phi(x,z,t) = 0$$

$$\text{Consider and periodic waveform:}\quad \Phi(x,z,t) = Z(z)\cos\big(k(x-ct)\big)$$

$$\Rightarrow \left(\frac{d^2}{dt^2} - k^2\right)Z(z) = 0$$

$$\text{Boundary condition at bottom of tank:}\quad v_z(x,0) = 0$$

$$\Rightarrow \frac{dZ}{dz}(0) = 0 \qquad\qquad Z(z) = A\cosh(kz)$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$\text{At surface:} \quad z = h + \zeta \qquad \frac{\partial \zeta}{\partial t} = v_z\left(x, h+\zeta, t\right) = -\frac{\partial \Phi\left(x, h+\zeta, t\right)}{\partial z}$$

$$-\frac{\partial \Phi\left(x, h+\zeta, t\right)}{\partial t} + g\zeta = 0$$

$$-\frac{\partial^2 \Phi\left(x, h+\zeta, t\right)}{\partial t^2} + g\frac{\partial \zeta}{\partial t} = -\frac{\partial^2 \Phi\left(x, h+\zeta, t\right)}{\partial t^2} - g\frac{\partial \Phi\left(x, h+\zeta, t\right)}{\partial z} = 0$$

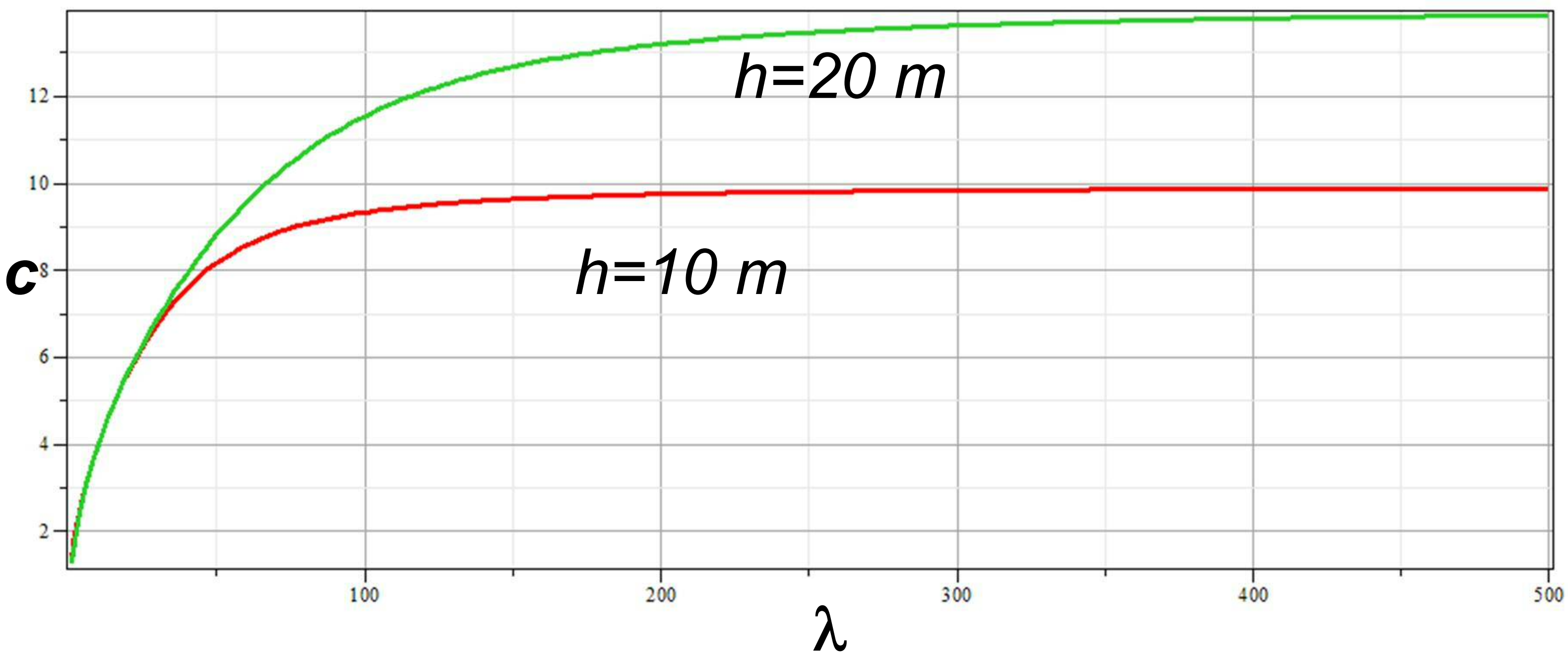
$$\text{For} \qquad \Phi(x, (h+\zeta), t) = A\cosh(k(h+\zeta))\cos(k(x-ct))$$

$$A\cosh(k(h+\zeta))\cos(k(x-ct))\left(k^2c^2 - gk\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}\right) = 0$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 = \frac{g}{k}\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}$$

Assuming $\zeta << h$: $$c^2 = \frac{g}{k}\tanh(kh)$$



For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 \approx \frac{g}{k}\tanh(kh) \qquad \text{For } \lambda >> h,\ c^2 \approx gh$$

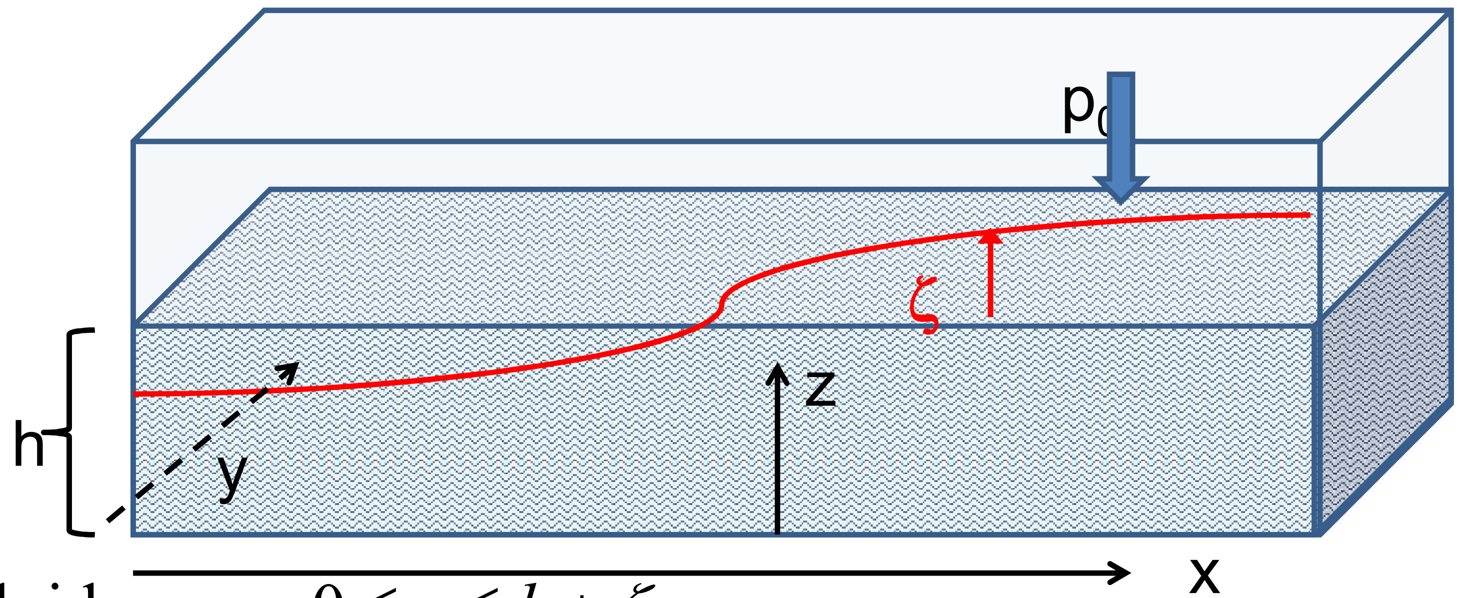
$$\Phi(x,z,t) = A\cosh(kz)\cos(k(x-ct))$$

$$\zeta(x,t) = \frac{1}{g}\frac{\partial\Phi(x,h+\zeta,t)}{\partial t} \approx \frac{kc}{g}A\cosh(kh)\sin(k(x-ct))$$

Note that for $\lambda >> h, \quad c^2 \approx gh$

(solutions are consistent with previous analysis)

General problem:



$$\text{Within fluid:} \qquad 0 \le z \le h + \zeta$$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2} v^2 + g(z-h) = \text{constant} \quad \text{(We have absorbed}$$

$$-\nabla^2 \Phi = 0 \qquad\qquad p_0 \text{ in our constant.)}$$

$$\text{At surface:} \quad z = h + \zeta \qquad \text{with } \zeta = \zeta(x, y, t)$$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} \qquad \text{where } v_{x,y} = v_{x,y}(x, y, h+\zeta, t)$$

Some links:

Details of soliton equations

Maple animation

Website – http://www.ma.hw.ac.uk/solitons/