

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF Olin 103**

Plan for Lecture 34:

Wave equation for sound

- 1. Non-linear effects in traveling sound wave**
- 2. Shock wave**

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22	Wed, 10/17/2012	Chap. 7, 5	Moment of inertia	
	Fri, 10/19/2012		Fall break	
23	Mon, 10/22/2012	Chap. 5	Rigid body rotation	#16
24	Wed, 10/24/2012	Chap. 5	Rigid body rotation	#17
25	Fri, 10/26/2012	Chap. 5	Rigid body rotation	#18
26	Mon, 10/29/2012	Chap. 8	Waves in elastic membranes	#19
27	Wed, 10/31/2012	Chap. 9	Introduction to hydrodynamics	
28	Fri, 11/01/2012	Chap. 9	Introduction to hydrodynamics	
29	Mon, 11/05/2012	Chap. 9	Introduction to hydrodynamics	#20
30	Wed, 11/07/2012	Chap. 9	Sound waves	
31	Fri, 11/09/2012	Chap. 9	Linear sound waves	#21
32	Mon, 11/12/2012	Chap. 9	Green's function for linear sound	
33	Wed, 11/14/2012	Chap. 9	Non-linear sound	
34	Fri, 11/16/2012	Chap. 9	Non-linear sound	Take Home Exam
35	Mon, 11/19/2012	Chap. 10	Surface waves	Take Home Exam
	Wed, 11/21/2012		Thanksgiving Holiday	
	Fri, 11/23/2012		Thanksgiving Holiday	
36	Mon, 11/26/2012	Chap. 10	Surface waves	Exam due
37	Wed, 11/28/2012	Chap. 10	Surface waves	
38	Fri, 11/30/2012	Chap. 10	Surface waves	
39	Mon, 12/03/2012		Student presentations I	
40	Wed, 12/05/2012		Student presentations II	

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**Effects of nonlinearities in fluid equations
-- one dimensional case**

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$

Assume that spatial variation confined to x direction ;
assume that $\mathbf{v} = v \hat{\mathbf{x}}$ and $\mathbf{f}_{\text{applied}} = 0$.

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0$$

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$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0$$

Expressing p in terms of ρ : $p = p(\rho)$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \rho} \frac{\partial \rho}{\partial x} \equiv c^2(\rho) \frac{\partial \rho}{\partial x} \quad \text{where} \quad \frac{\partial p}{\partial \rho} \equiv c^2(\rho)$$

For adiabatic ideal gas :

$$c^2(\rho) = \frac{\mathcal{P}}{\rho} = c_0^2 \left(\frac{\rho}{\rho_0} \right)^{\gamma-1} \quad \text{where} \quad c_0^2 \equiv \frac{\mathcal{P}_0}{\rho_0}$$

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$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{c^2(\rho)}{\rho} \frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0$$

Expressing variation v in terms of variation of ρ :

$$\frac{\partial v}{\partial \rho} \frac{\partial \rho}{\partial t} + v \frac{\partial v}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{c^2(\rho)}{\rho} \frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial \rho} \frac{\partial \rho}{\partial x} = 0$$

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Some more algebra :

From Euler equation : $\frac{\partial v}{\partial \rho} \left(\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} \right) + \frac{c^2(\rho)}{\rho} \frac{\partial \rho}{\partial x} = 0$

From continuity equation : $\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} = -\rho \frac{\partial v}{\partial \rho} \frac{\partial \rho}{\partial x}$

Combined equation : $\frac{\partial v}{\partial \rho} \left(-\rho \frac{\partial v}{\partial \rho} \frac{\partial \rho}{\partial x} \right) + \frac{c^2(\rho)}{\rho} \frac{\partial \rho}{\partial x} = 0$

$$\Rightarrow \left(\frac{\partial v}{\partial \rho} \right)^2 = \frac{c^2(\rho)}{\rho^2} \quad \frac{\partial v}{\partial \rho} = \pm \frac{c}{\rho}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + (v \pm c) \frac{\partial \rho}{\partial x} = 0$$

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Assuming adiabatic process : $c^2 = c_0^2 \left(\frac{\rho}{\rho_0} \right)^{\gamma-1}$ $c_0^2 = \frac{\gamma P_0}{\rho_0}$

$$\frac{\partial v}{\partial \rho} = \pm \frac{c}{\rho} \Rightarrow v = \pm c_0 \int_{\rho_0}^{\rho} \left(\frac{\rho'}{\rho_0} \right)^{(\gamma-1)/2} \frac{d\rho'}{\rho'}$$

$$\Rightarrow v = \pm \frac{2c_0}{\gamma-1} \left(\left(\frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} - 1 \right)$$

$$\Rightarrow c = c_0 \left(\frac{\rho}{\rho_0} \right)^{(\gamma-1)/2}$$

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Traveling wave solution:

Assume : $\rho = \rho_0 + f(x - u(\rho)t)$

Need to find self-consistent equations for propagation velocity $u(\rho)$ using equations

From previous derivations : $\frac{\partial \rho}{\partial t} + (v \pm c) \frac{\partial \rho}{\partial x} = 0$

Apparently : $u(\rho) \Leftrightarrow v \pm c$

For adiabatic ideal gas and + signs :

$$u = v + c = c_0 \left(\frac{\gamma+1}{\gamma-1} \left(\frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} - \frac{2}{\gamma-1} \right)$$

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Visualization for particular waveform : $\rho = \rho_0 + f(x - u(\rho)t)$

Assume : $f(w) \equiv f_0 s(w)$

$$\frac{\rho}{\rho_0} = 1 + \frac{f_0}{\rho_0} s(x - ut)$$

For adiabatic ideal gas :

$$u = c_0 \left(\frac{\gamma+1}{\gamma-1} \left(\frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} - \frac{2}{\gamma-1} \right)$$

$$u = c_0 \left(\frac{\gamma+1}{\gamma-1} \left(1 + \frac{f_0}{\rho_0} s(x - ut) \right)^{(\gamma-1)/2} - \frac{2}{\gamma-1} \right)$$

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Visualization continued:

$$u = c_0 \left(\frac{\gamma+1}{\gamma-1} \left(1 + \frac{f_0}{\rho_0} s(x-ut) \right)^{(\gamma-1)/2} - \frac{2}{\gamma-1} \right)$$

Plot $s(x-ut)$ for fixed t , as a function of x :

Let $w = x - ut$

$$x = w + ut = w + u(w)t$$

$$u(w) = c_0 \left(\frac{\gamma+1}{\gamma-1} \left(1 + \frac{f_0}{\rho_0} s(w) \right)^{(\gamma-1)/2} - \frac{2}{\gamma-1} \right)$$

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