

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF Olin 103**

Plan for Lecture 31:

Wave equation for sound

- 1. Standing waves**
- 2. Green's function for wave equation; wave scattering**

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|----|-----------------|-------------|-------------------------------|----------------|
| 8 | Fri, 9/14/2012 | Chap. 3 | Lagrangian | #7 |
| 9 | Mon, 9/17/2012 | Chap. 3 & 6 | Lagrangian | #8 |
| 10 | Wed, 9/19/2012 | Chap. 3 & 6 | Lagrangian | #9 |
| 11 | Fri, 9/21/2012 | Chap. 3 & 6 | Lagrangian | #10 |
| 12 | Mon, 9/24/2012 | Chap. 3 & 6 | Lagrangian and Hamiltonian | #11 |
| 13 | Wed, 9/26/2012 | Chap. 6 | Lagrangian and Hamiltonian | #12 |
| 14 | Fri, 9/28/2012 | Chap. 6 | Lagrangian and Hamiltonian | #13 |
| 15 | Mon, 10/01/2012 | Chap. 4 | Small oscillations | #14 |
| 16 | Wed, 10/03/2012 | Chap. 4 | Small oscillations | #15 |
| 17 | Fri, 10/05/2012 | Chap. 4 | Small oscillations | |
| 18 | Mon, 10/08/2012 | Chap. 7 | Wave equation | Take Home Exam |
| 19 | Wed, 10/10/2012 | Chap. 7 | Wave equation | Take Home Exam |
| 20 | Fri, 10/12/2012 | Chap. 7 | Wave equation | Take Home Exam |
| 21 | Mon, 10/15/2012 | Chap. 7 | Wave equation | Exam due |
| 22 | Wed, 10/17/2012 | Chap. 7, 5 | Moment of inertia | |
| | Fri, 10/19/2012 | | Fall break | |
| 23 | Mon, 10/22/2012 | Chap. 5 | Rigid body rotation | #16 |
| 24 | Wed, 10/24/2012 | Chap. 5 | Rigid body rotation | #17 |
| 25 | Fri, 10/26/2012 | Chap. 5 | Rigid body rotation | #18 |
| 26 | Mon, 10/29/2012 | Chap. 8 | Waves in elastic membranes | #19 |
| 27 | Wed, 10/31/2012 | Chap. 9 | Introduction to hydrodynamics | |
| 28 | Fri, 11/01/2012 | Chap. 9 | Introduction to hydrodynamics | |
| 29 | Mon, 11/05/2012 | Chap. 9 | Introduction to hydrodynamics | #20 |
| 30 | Wed, 11/07/2012 | Chap. 9 | Sound waves | |
| 31 | Fri, 11/09/2012 | Chap. 9 | Linear sound waves | #21 |

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Linearization of the fluid dynamics relations:

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

$$\text{Continuity equation : } \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Near equilibrium m :

$$\rho = \rho_0 + \delta \rho$$

$$p = p_0 + \delta p$$

$$\mathbf{v} = 0 + \delta \mathbf{v}$$

$$\mathbf{f}_{\text{applied}} = 0$$

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Equations to lowest order in perturbation :

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0}$$

$$\frac{\partial \delta \rho}{\partial t} + \rho_0 \nabla \cdot \delta \mathbf{v} = 0$$

Velocity potential : $\delta \mathbf{v} = -\nabla \Phi$

Pressure in terms of the density :

$$\delta p = \left(\frac{\partial p}{\partial \rho} \right)_{s, \rho_0, p_0} \delta \rho \equiv c^2 \delta \rho$$

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

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Wave equation for air :

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho} \right)_s = \frac{\mathcal{P}_0}{\rho_0}$

$\mathbf{v} = -\nabla \Phi$

Boundary values :

Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity \mathbf{V} :

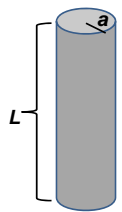
$$\hat{\mathbf{n}} \cdot \mathbf{V} = \hat{\mathbf{n}} \cdot \delta \mathbf{v} = -\hat{\mathbf{n}} \cdot \nabla \Phi$$

Free surface :

$$\delta p = 0 \quad \Rightarrow \quad \rho_0 \frac{\partial \Phi}{\partial t} = 0$$

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Time harmonic standing waves in a pipe



$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Boundary values :

At fixed surface : $\hat{\mathbf{n}} \cdot \nabla \Phi = 0$

At free surface : $\frac{\partial \Phi}{\partial t} = 0$

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$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0 \quad \text{Define : } k \equiv \frac{\omega}{c}$$

In cylindrical coordinates :

$$\Phi(r, \varphi, z, t) = R(r)F(\varphi)Z(z)e^{-i\omega t}$$

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

$$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \Phi(r, \varphi, z, t) = 0$$

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$$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \Phi(r, \varphi, z, t) = 0$$

$$\Phi(r, \varphi, z, t) = R(r)F(\varphi)Z(z)e^{-i\omega t}$$

$F(\varphi) = e^{im\varphi}$; $F(\varphi) = F(\varphi + 2\pi N) \Rightarrow m = \text{integer}$

$Z(z) = e^{i\alpha z}$; $\alpha = \text{real plus other restrictions}$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} - \alpha^2 + k^2 \right) R(r) = 0$$

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$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} - \alpha^2 + k^2 \right) R(r) = 0$$

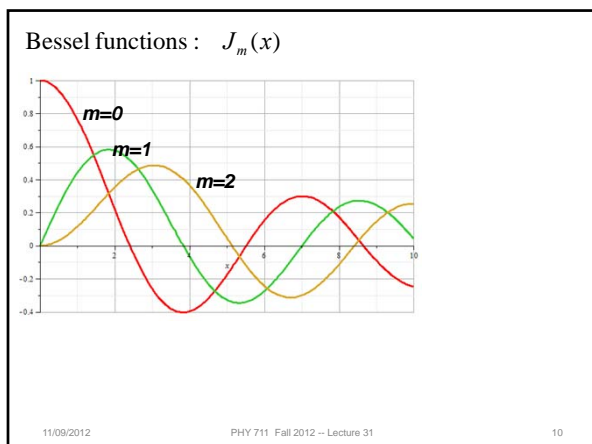
For $k^2 \geq \alpha^2$ define $\kappa^2 \equiv k^2 - \alpha^2$

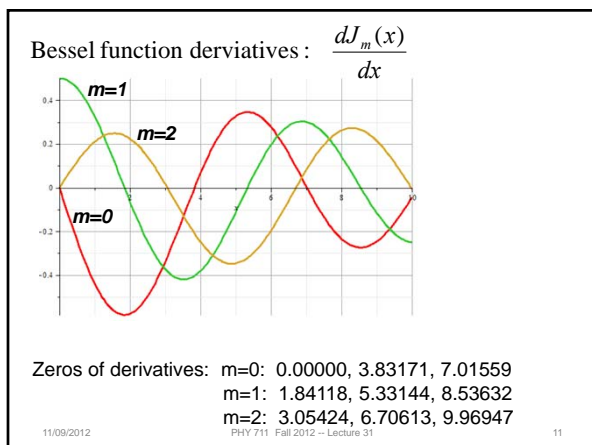
$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} + \kappa^2 \right) R(r) = 0$$

Cylinder surface boundary conditions : $\left. \frac{dR}{dr} \right|_{r=a} = 0$

$\Rightarrow R(r) = J_m(\kappa r)$ where for $\frac{dJ_m(x'_{mn})}{dx} = 0$, $\kappa_{mn} = \frac{x'_{mn}}{a}$

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Boundary condition for $z=0, z=L$:

For open - open pipe :

$$Z(0) = Z(L) = 0 \Rightarrow Z(z) = \sin\left(\frac{p\pi z}{L}\right)$$

$$\Rightarrow \alpha_p = \frac{p\pi}{L}, \quad p = 1, 2, 3, \dots$$

Resonant frequencies :

$$\frac{\omega^2}{c^2} = k^2 = \kappa_{mn}^2 + \alpha_p^2$$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{p\pi}{L}\right)^2$$

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Example

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2 = \left(\frac{\pi p}{L}\right)^2 \left(1 + \left(\frac{L}{a}\right)^2 \left(\frac{x'_{mn}}{\pi p}\right)^2\right)$$

$$\pi p = 3.14, 6.28, 9.42, \dots$$

$$x'_{mn} = 0.00, 1.84, 3.05$$

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Alternate boundary condition for $z=0, z=L$:

For open - closed pipe :

$$\frac{dZ(0)}{dz} = Z(L) = 0 \Rightarrow Z(z) = \cos\left(\frac{(2p+1)\pi z}{2L}\right)$$

$$\Rightarrow \alpha_p = \frac{(2p+1)\pi}{2L}, \quad p = 0, 1, 2, 3, \dots$$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi(2p+1)}{2L}\right)^2$$

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Other solutions to wave equation:

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0$$

Plane wave solution :

$$\Phi(\mathbf{r}, t) = A e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c}\right)^2$$

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Wave equation with source:

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -f(\mathbf{r}, t)$$

Solution in terms of Green's function :

$$\Phi(\mathbf{r}, t) = \int d^3r' \int dt' G(\mathbf{r} - \mathbf{r}', t - t') f(\mathbf{r}', t')$$

where

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(\mathbf{r} - \mathbf{r}', t - t') = -\delta(\mathbf{r} - \mathbf{r}') \delta(t - t')$$

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Wave equation with source -- continued:

We can show that :

$$G(\mathbf{r} - \mathbf{r}', t - t') = \frac{\delta\left(t' - \left(t \mp \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

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