PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 28:

Introduction to hydrodynamics

1. Correction – Euler formulation revisited
2. Euler's equation for fluid dynamics
3. Bernoulli's integrals

7	Wed, 9/12/2012	Chap. 3	Calculus of Variation continued	
8	Fri, 9/14/2012	Chap. 3	Lagrangian	#7
9	Mon, 9/17/2012	Chap. 3 & 6	Lagrangian	#8
10	Wed, 9/19/2012	Chap. 3 & 6	Lagrangian	#9
11	Fri, 9/21/2012	Chap. 3 & 6	Lagrangian	#10
12	Mon, 9/24/2012	Chap. 3 & 6	Lagrangian and Hamiltonian	#11
13	Wed, 9/26/2012	Chap. 6	Lagrangian and Hamiltonian	#12
14	Fri, 9/28/2012	Chap. 6	Lagrangian and Hamiltonian	#13
15	Mon, 10/01/2012	Chap. 4	Small oscillations	#14
16	Wed, 10/03/2012	Chap. 4	Small oscillations	#15
17	Fri, 10/05/2012	Chap. 4	Small oscillations	
18	Mon, 10/08/2012	Chap. 7	Wave equation	Take Home Exam
19	Wed, 10/10/2012	Chap. 7	Wave equation	Take Home Exam
20	Fri, 10/12/2012	Chap. 7	Wave equation	Take Home Exam
21	Mon, 10/15/2012	Chap. 7	Wave equation	Exam due
22	Wed, 10/17/2012	Chap. 7, 5	Moment of inertia	
	Fri, 10/19/2012		Fall break	
23	Mon, 10/22/2012	Chap. 5	Rigid body rotation	#16
24	Wed, 10/24/2012	Chap. 5	Rigid body rotation	#17
25	Fri, 10/26/2012	Chap. 5	Rigid body rotation	#18
26	Mon, 10/29/2012	Chap. 8	Waves in elastic membranes	#19
27	Wed, 10/31/2012	Chap. 9	Introduction to hydrodynamics	
28	Fri, 11/01/2012	Chap. 9	Introduction to hydrodynamics	

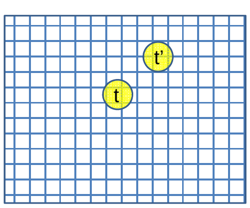
Newton's equations for fluids

Use **Euler** formulation; properties described in terms of stationary spatial grid

Variables: Density $\rho(x,y,z,t)$

Pressure $p(x,y,z,t)$

Velocity $\mathbf{v}(x,y,z,t)$



Particle at t: $\mathbf{r}, t$

Particle at t': $\mathbf{r} + \mathbf{v}\delta t, t'$

Euler analysis -- continued

Particle at t: $\mathbf{r}, t$

Particle at t': $\mathbf{r}+\mathbf{v}\delta t, t'$ where $\delta t = t' - t$

For $f(\mathbf{r},t)$:

$$\frac{df}{dt} = \lim_{\delta t\to 0}\left(\frac{f(\mathbf{r},t')-f(\mathbf{r},t)}{\delta t}+\frac{f(\mathbf{r}+\mathbf{v}\delta t,t)-f(\mathbf{r},t)}{\delta t}\right)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v}\cdot\nabla)f$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho\nabla\cdot\mathbf{v} + \nabla\rho\cdot\mathbf{v} = 0$$

Consider : $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \nabla\rho\cdot\mathbf{v}$

$\Rightarrow \frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0$ alternative form of continuity equation

Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times(\nabla\times\mathbf{v}) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $(\nabla\times\mathbf{v}) = 0$ "irrotational flow"

   $\Rightarrow \mathbf{v} = -\nabla\Phi$

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force

3. $\rho =$ (constant) incompressible fluid

$$\frac{\partial(-\nabla\Phi)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Bernoulli's integral of Euler's equation for constant ρ

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Integrating over space:

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C(t)$$

where $\mathbf{v}=-\nabla\Phi(\mathbf{r},t)=-\nabla\left(\Phi(\mathbf{r},t)+C(t)\right)$

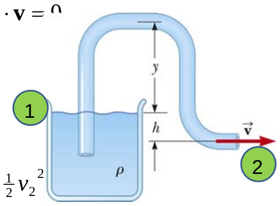
$$\Rightarrow\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C_0 \qquad \text{Bernoulli's theorem}$$

Examples of Bernoulli's theorem for constant ρ

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C_0$$

For steady flow: $\frac{\partial\Phi}{\partial t}=0$

Continuity equation: $\nabla\cdot\mathbf{v}=0$



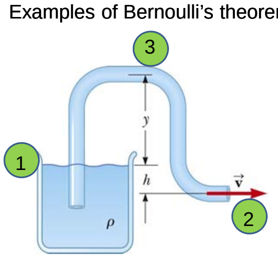
$p_1=p_2=p_{atm}$

$U_1-U_2=gh$

$v_1\approx 0$

$$\frac{p_1}{\rho}+U_1+\tfrac{1}{2}v_1^2=\frac{p_2}{\rho}+U_2+\tfrac{1}{2}v_2^2$$

Examples of Bernoulli's theorem -- continued



$p_1=p_2=p_{atm}$

$U_1-U_2=gh$

$v_1\approx 0 \qquad (v_1A_1=v_2A_2)$

$$\frac{p_1}{\rho}+U_1+\tfrac{1}{2}v_1^2=\frac{p_2}{\rho}+U_2+\tfrac{1}{2}v_2^2$$

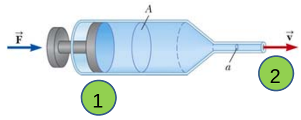
$v_2\approx\sqrt{2gh}$

$p_3=0; \qquad U_3-U_1=gy \qquad v_3=\sqrt{2p_{atm}/\rho-2gy}$

$$\frac{p_1}{\rho}+U_1+\tfrac{1}{2}v_1^2=\frac{p_3}{\rho}+U_3+\tfrac{1}{2}v_3^2 \Rightarrow y\le\frac{p_{atm}}{\rho g}=\frac{1.013\times10^5}{(1000)(9.8)}m =10.3m$$

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$



$$p_1 = \frac{F}{A} + p_{atm} \qquad p_2 = p_{atm}$$

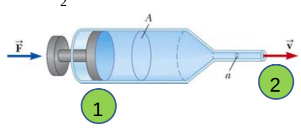
$$U_1 = U_2$$

$$v_1 A = v_2 a \qquad \text{continuity equation}$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}{v_1}^2 = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}{v_2}^2$$

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$



$$\frac{2F}{\rho A} = {v_2}^2\left(1 - \left(\frac{a}{A}\right)^2\right)$$

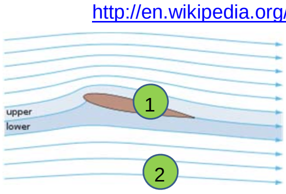
$$v_2 = \sqrt{\frac{2F/(\rho A)}{1 - \left(\frac{a}{A}\right)^2}}$$

Examples of Bernoulli's theorem – continued

Approximate explanation of airplane lift

Cross section view of airplane wing

http://en.wikipedia.org/wiki/Lift_%28force%29



$$U_1 \approx U_2$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}{v_1}^2 = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}{v_2}^2$$

$$p_2 - p_1 = \tfrac{1}{2}\rho\left({v_1}^2 - {v_2}^2\right)$$

Some details on the velocity potential

Continuity equation :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} + \nabla \rho \cdot \mathbf{v} = 0$$

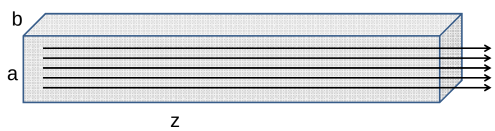
For incompressible fluid : $\rho = (\text{constant})$

$$\Rightarrow \nabla \cdot \mathbf{v} = 0$$

Irrotational flow : $\nabla \times \mathbf{v} = 0 \qquad \Rightarrow \mathbf{v} = -\nabla \Phi$

$$\Rightarrow \nabla^2 \Phi = 0$$

Example – uniform flow



$$\nabla^2 \Phi = 0$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Possible solution :

$$\Phi = -v_o z$$

$$\mathbf{v} = -\nabla \Phi = v_o \hat{\mathbf{z}}$$

Example – flow around a cylinder

$$\nabla^2 \Phi = 0$$

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$$

to be continued…

Solution of Euler's equation for fluids -- isentropic

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $(\nabla\times\mathbf{v}) = 0$ "irrotational flow"
   $\Rightarrow \mathbf{v} = -\nabla\Phi$
2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force
3. $\rho \neq$ (constant) isentropic fluid

A little thermodynamics

First law of thermodynamics: $dE_{\text{int}} = dQ - dW$

For isentropic conditions: $dQ = 0$

$dE_{\text{int}} = -dW = -pdV$

Solution of Euler's equation for fluids – isentropic (continued)

$dE_{\text{int}} = -dW = pdV$

In terms of mass density: $\rho = \frac{M}{V}$

For fixed M and variable V: $d\rho = -\frac{M}{V^2}dV$

$$dV = -\frac{M}{\rho^2}d\rho$$

In terms in intensive variables: Let $E_{\text{int}} = M\varepsilon$

$$dE_{\text{int}} = Md\varepsilon = -dW = -pdV = M\frac{p}{\rho^2}d\rho$$

$$d\varepsilon = \frac{p}{\rho^2}d\rho \qquad \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

Solution of Euler's equation for fluids – isentropic (continued)

$$\left(\frac{\partial\varepsilon}{\partial\rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

Consider: $\nabla\varepsilon = \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{dQ=0}\nabla\rho = \frac{p}{\rho^2}\nabla\rho$

Rearranging: $\nabla\left(\varepsilon + \frac{p}{\rho}\right) = \frac{\nabla p}{\rho}$

Solution of Euler's equation for fluids – isentropic (continued)

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times(\nabla\times\mathbf{v}) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

$$\frac{\nabla p}{\rho} = \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\nabla\times\mathbf{v} = 0 \qquad \mathbf{v} = -\nabla\Phi \qquad \mathbf{f}_{applied} = -\nabla U$$

$$\frac{\partial(-\nabla\Phi)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\Rightarrow \nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$