

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF Olin 103**

Plan for Lecture 28:

Introduction to hydrodynamics

- 1. Correction – Euler formulation revisited**
- 2. Euler's equation for fluid dynamics**
- 3. Bernoulli's integrals**

7	Wed, 9/12/2012	Chap. 3	Calculus of Variation continued	
8	Fri, 9/14/2012	Chap. 3	Lagrangian	#7
9	Mon, 9/17/2012	Chap. 3 & 6	Lagrangian	#8
10	Wed, 9/19/2012	Chap. 3 & 6	Lagrangian	#9
11	Fri, 9/21/2012	Chap. 3 & 6	Lagrangian	#10
12	Mon, 9/24/2012	Chap. 3 & 6	Lagrangian and Hamiltonian	#11
13	Wed, 9/26/2012	Chap. 6	Lagrangian and Hamiltonian	#12
14	Fri, 9/28/2012	Chap. 6	Lagrangian and Hamiltonian	#13
15	Mon, 10/01/2012	Chap. 4	Small oscillations	#14
16	Wed, 10/03/2012	Chap. 4	Small oscillations	#15
17	Fri, 10/05/2012	Chap. 4	Small oscillations	
18	Mon, 10/08/2012	Chap. 7	Wave equation	Take Home Exam
19	Wed, 10/10/2012	Chap. 7	Wave equation	Take Home Exam
20	Fri, 10/12/2012	Chap. 7	Wave equation	Take Home Exam
21	Mon, 10/15/2012	Chap. 7	Wave equation	Exam due
22	Wed, 10/17/2012	Chap. 7, 5	Moment of inertia	
	Fri, 10/19/2012		Fall break	
23	Mon, 10/22/2012	Chap. 5	Rigid body rotation	#16
24	Wed, 10/24/2012	Chap. 5	Rigid body rotation	#17
25	Fri, 10/26/2012	Chap. 5	Rigid body rotation	#18
26	Mon, 10/29/2012	Chap. 8	Waves in elastic membranes	#19
27	Wed, 10/31/2012	Chap. 9	Introduction to hydrodynamics	
28	Fri, 11/01/2012	Chap. 9	Introduction to hydrodynamics	



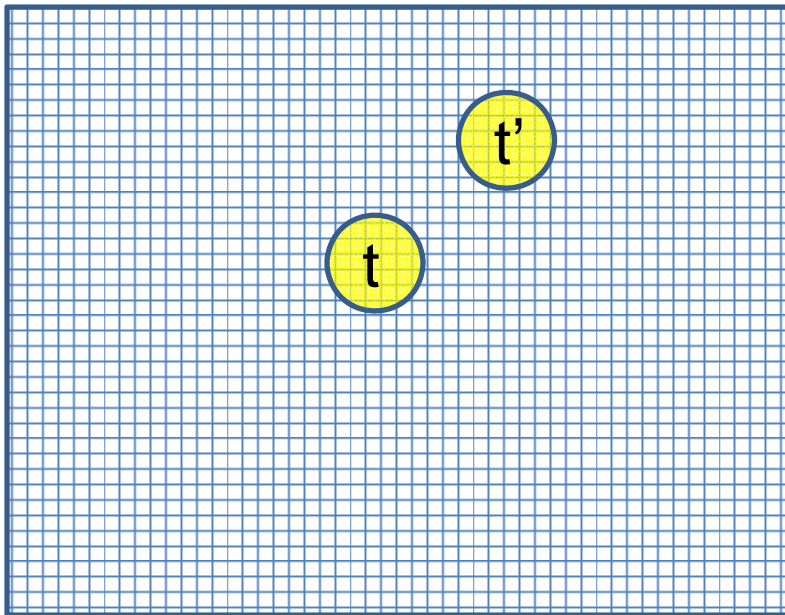
Newton's equations for fluids

Use **Euler** formulation; properties described in terms of stationary spatial grid

Variables : Density $\rho(x, y, z, t)$

Pressure $p(x, y, z, t)$

Velocity $\mathbf{v}(x, y, z, t)$



Particle at t : \mathbf{r}, t

Particle at t' : $\mathbf{r} + \mathbf{v}\delta t, t'$

Euler analysis -- continued

Particle at t : \mathbf{r}, t

Particle at t' : $\mathbf{r} + \mathbf{v}\delta t, t'$ where $\delta t = t' - t$

For $f(\mathbf{r}, t)$:

$$\frac{df}{dt} = \lim_{\delta t \rightarrow 0} \left(\frac{f(\mathbf{r}, t') - f(\mathbf{r}, t)}{\delta t} + \frac{f(\mathbf{r} + \mathbf{v}\delta t, t) - f(\mathbf{r}, t)}{\delta t} \right)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla) f$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} + \nabla \rho \cdot \mathbf{v} = 0$$

Consider : $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \nabla \rho \cdot \mathbf{v}$

$$\Rightarrow \frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0$$

alternative form

of continuity equation

Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

Consider the following restrictions :

1. $(\nabla \times \mathbf{v}) = 0$ "irrotational flow"

$$\Rightarrow \mathbf{v} = -\nabla \Phi$$

2. $\mathbf{f}_{\text{applied}} = -\nabla U$ conservative applied force

3. $\rho = (\text{constant})$ incompressible fluid

$$\frac{\partial(-\nabla \Phi)}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla \left(\frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} \right) = 0$$

Bernoulli's integral of Euler's equation for constant ρ

$$\nabla \left(\frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} \right) = 0$$

Integrating over space:

$$\frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} = C(t)$$

where $\mathbf{v} = -\nabla \Phi(\mathbf{r}, t) = -\nabla(\Phi(\mathbf{r}, t) + C(t))$

$$\Rightarrow \frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} = C_0 \quad \text{Bernoulli's theorem}$$

Examples of Bernoulli's theorem for constant ρ

$$\frac{p}{\rho} + U + \frac{1}{2}v^2 - \frac{\partial\Phi}{\partial t} = C_0$$

For steady flow : $\frac{\partial\Phi}{\partial t} = 0$

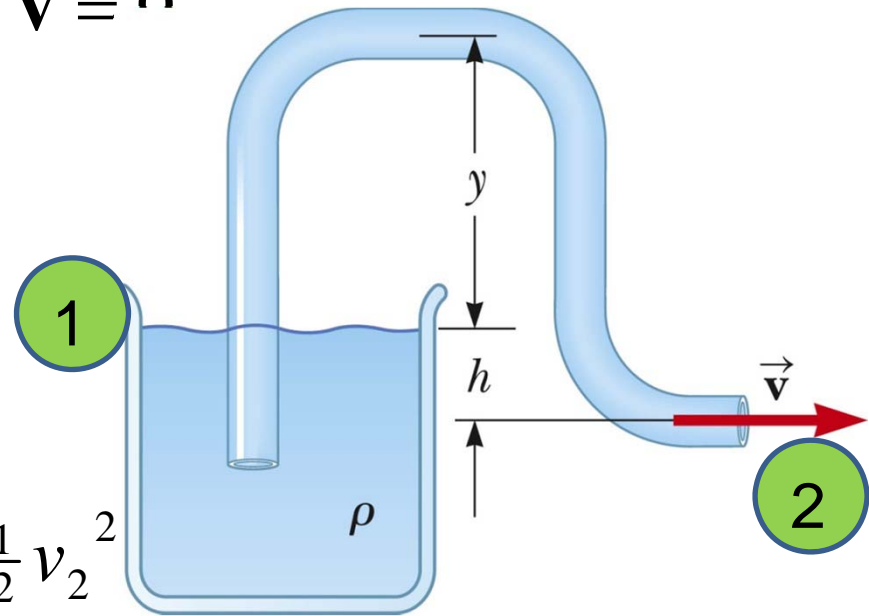
Continuity equation : $\nabla \cdot \mathbf{v} = 0$

$$p_1 = p_2 = p_{atm}$$

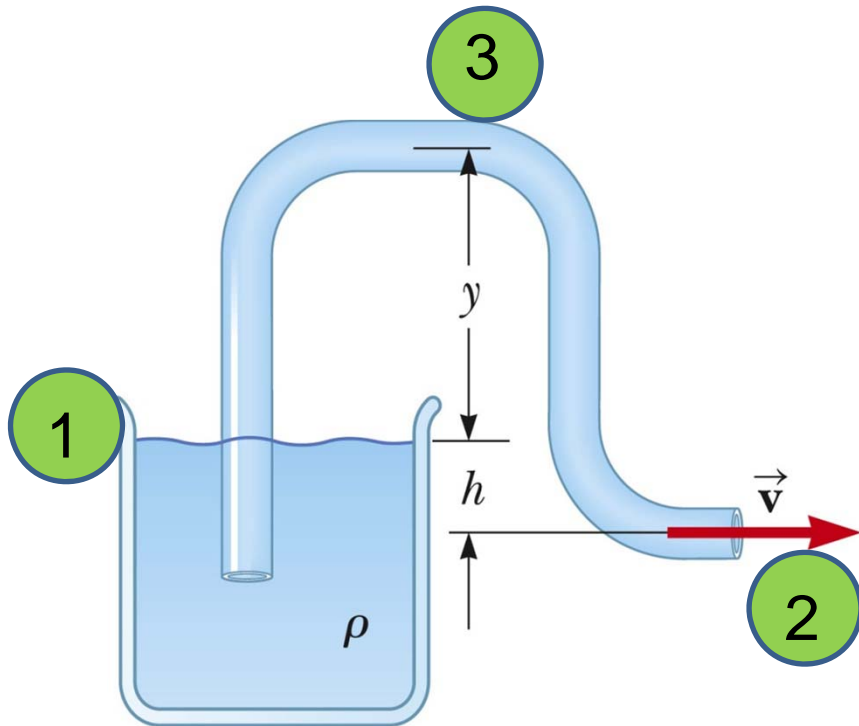
$$U_1 - U_2 = gh$$

$$v_1 \approx 0$$

$$\frac{p_1}{\rho} + U_1 + \frac{1}{2}v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2}v_2^2$$



Examples of Bernoulli's theorem -- continued



$$p_1 = p_2 = p_{atm}$$

$$U_1 - U_2 = gh$$

$$v_1 \approx 0 \quad (v_1 A_1 = v_2 A_2)$$

$$\frac{p_1}{\rho} + U_1 + \frac{1}{2} v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2} v_2^2$$

$$v_2 \approx \sqrt{2gh}$$

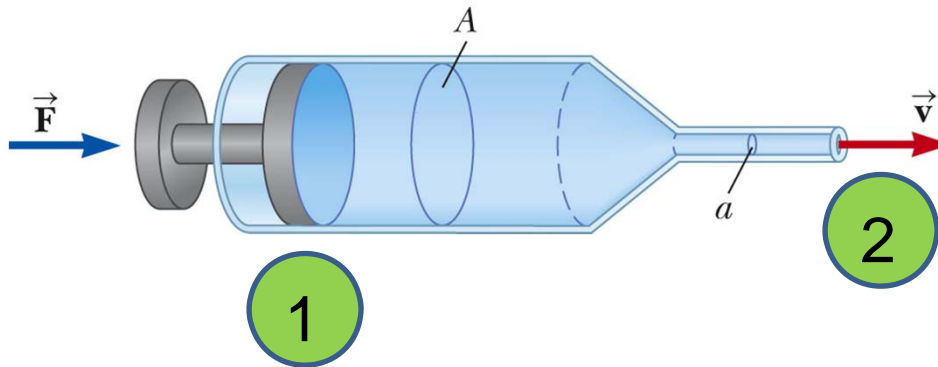
$$p_3 = 0; \quad U_3 - U_1 = gy$$

$$v_3 = \sqrt{2p_{atm} / \rho - 2gy}$$

$$\frac{p_1}{\rho} + U_1 + \frac{1}{2} v_1^2 = \frac{p_3}{\rho} + U_3 + \frac{1}{2} v_3^2 \Rightarrow y \leq \frac{p_{atm}}{\rho g} = \frac{1.013 \times 10^5}{(1000)(9.8)} m = 10.3 m$$

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \frac{1}{2} v^2 = \text{constant}$$



$$p_1 = \frac{F}{A} + p_{atm} \quad p_2 = p_{atm}$$

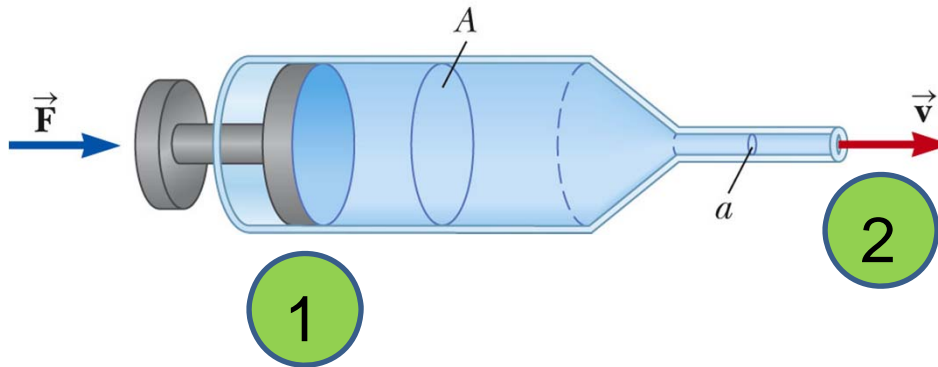
$$U_1 = U_2$$

$$v_1 A = v_2 a \quad \text{continuity equation}$$

$$\frac{p_1}{\rho} + U_1 + \frac{1}{2} v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2} v_2^2$$

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \frac{1}{2} v^2 = \text{constant}$$



$$\frac{2F}{\rho A} = v_2^2 \left(1 - \left(\frac{a}{A} \right)^2 \right)$$

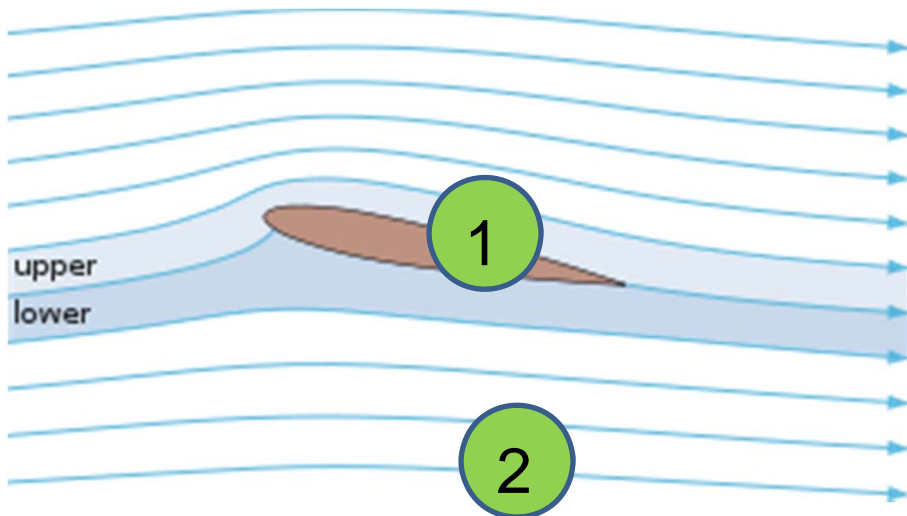
$$v_2 = \sqrt{\frac{2F / (\rho A)}{1 - \left(\frac{a}{A} \right)^2}}$$

Examples of Bernoulli's theorem – continued

Approximate explanation of airplane lift

Cross section view of airplane wing

http://en.wikipedia.org/wiki/Lift_%28force%29



$$U_1 \approx U_2$$

$$\frac{p_1}{\rho} + U_1 + \frac{1}{2} v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2} v_2^2$$

$$p_2 - p_1 = \frac{1}{2} \rho (v_1^2 - v_2^2)$$

Some details on the velocity potential

Continuity equation :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} + \nabla \rho \cdot \mathbf{v} = 0$$

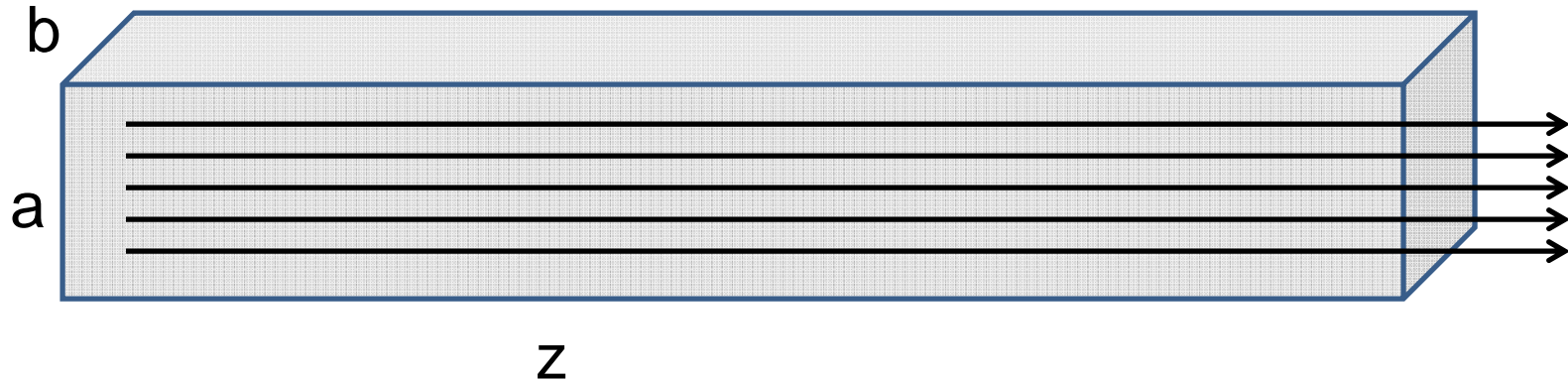
For incompressible fluid : $\rho = (\text{constant})$

$$\Rightarrow \nabla \cdot \mathbf{v} = 0$$

Irrotational flow : $\nabla \times \mathbf{v} = 0 \quad \Rightarrow \mathbf{v} = -\nabla \Phi$

$$\Rightarrow \nabla^2 \Phi = 0$$

Example – uniform flow



$$\nabla^2 \Phi = 0$$

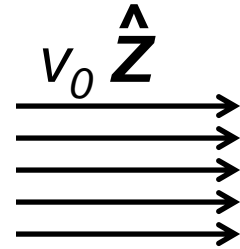
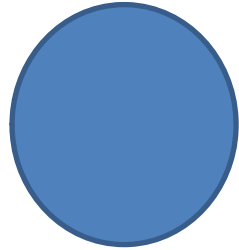
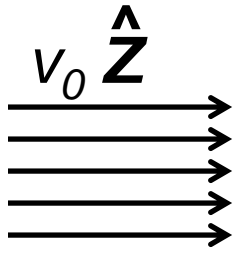
$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Possible solution :

$$\Phi = -v_o z$$

$$\mathbf{v} = -\nabla \Phi = v_o \hat{\mathbf{z}}$$

Example – flow around a cylinder



$$\nabla^2 \Phi = 0$$

$$\left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = 0$$

to be continued...

Solution of Euler's equation for fluids -- isentropic

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

Consider the following restrictions :

1. $(\nabla \times \mathbf{v}) = 0$ "irrotational flow"

$$\Rightarrow \mathbf{v} = -\nabla \Phi$$

2. $\mathbf{f}_{\text{applied}} = -\nabla U$ conservative applied force

3. $\rho \neq (\text{constant})$ isentropic fluid

A little thermodynamics

First law of thermodynamics : $dE_{\text{int}} = dQ - dW$

For isentropic conditions : $dQ = 0$

$$dE_{\text{int}} = -dW = -pdV$$

Solution of Euler's equation for fluids – isentropic (continued)

$$dE_{\text{int}} = -dW = pdV$$

In terms of mass density : $\rho = \frac{M}{V}$

For fixed M and variable V : $d\rho = -\frac{M}{V^2}dV$

$$dV = -\frac{M}{\rho^2}d\rho$$

In terms in intensive variables : Let $E_{\text{int}} = M\varepsilon$

$$dE_{\text{int}} = Md\varepsilon = -dW = -pdV = M \frac{p}{\rho^2} d\rho$$

$$d\varepsilon = \frac{p}{\rho^2} d\rho \quad \left(\frac{\partial \varepsilon}{\partial \rho} \right)_{dQ=0} = \frac{p}{\rho^2}$$

Solution of Euler's equation for fluids – isentropic (continued)

$$\left(\frac{\partial \varepsilon}{\partial \rho} \right)_{dQ=0} = \frac{p}{\rho^2}$$

Consider: $\nabla \varepsilon = \left(\frac{\partial \varepsilon}{\partial \rho} \right)_{dQ=0} \nabla \rho = \frac{p}{\rho^2} \nabla \rho$

Rearranging: $\nabla \left(\varepsilon + \frac{p}{\rho} \right) = \frac{\nabla p}{\rho}$

Solution of Euler's equation for fluids – isentropic (continued)

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

$$\frac{\nabla p}{\rho} = \nabla \left(\varepsilon + \frac{p}{\rho} \right)$$

$$\nabla \times \mathbf{v} = 0$$

$$\mathbf{v} = -\nabla \Phi$$

$$\mathbf{f}_{\text{applied}} = -\nabla U$$

$$\frac{\partial(-\nabla\Phi)}{\partial t} + \nabla\left(\frac{1}{2}v^2\right) = -\nabla U - \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\Rightarrow \nabla\left(\varepsilon + \frac{p}{\rho} + U + \frac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$