PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 26:

Traveling and standing waves in 1 and 2 dimensions

1. **Brief review of 1-d from Chap. 7**

2. **Waves in 2-d elastic membranes – Chap. 8**

10/29/2012　　　　　PHY 711 Fall 2012 -- Lecture 26　　　　　1

6	Mon, 9/10/2012	Chap. 3	Calculus of Variation	#6
7	Wed, 9/12/2012	Chap. 3	Calculus of Variation continued	
8	Fri, 9/14/2012	Chap. 3	Lagrangian	#7
9	Mon, 9/17/2012	Chap. 3 & 6	Lagrangian	#8
10	Wed, 9/19/2012	Chap. 3 & 6	Lagrangian	#9
11	Fri, 9/21/2012	Chap. 3 & 6	Lagrangian	#10
12	Mon, 9/24/2012	Chap. 3 & 6	Lagrangian and Hamiltonian	#11
13	Wed, 9/26/2012	Chap. 6	Lagrangian and Hamiltonian	#12
14	Fri, 9/28/2012	Chap. 6	Lagrangian and Hamiltonian	#13
15	Mon, 10/01/2012	Chap. 4	Small oscillations	#14
16	Wed, 10/03/2012	Chap. 4	Small oscillations	#15
17	Fri, 10/05/2012	Chap. 4	Small oscillations	
18	Mon, 10/08/2012	Chap. 7	Wave equation	Take Home Exam
19	Wed, 10/10/2012	Chap. 7	Wave equation	Take Home Exam
20	Fri, 10/12/2012	Chap. 7	Wave equation	Take Home Exam
21	Mon, 10/15/2012	Chap. 7	Wave equation	Exam due
22	Wed, 10/17/2012	Chap. 7, 5	Moment of inertia	
	Fri, 10/19/2012		Fall break	
23	Mon, 10/22/2012	Chap. 5	Rigid body rotation	#16
24	Wed, 10/24/2012	Chap. 5	Rigid body rotation	#17
25	Fri, 10/26/2012	Chap. 5	Rigid body rotation	#18
26	Mon, 10/29/2012	Chap. 8	Waves in elastic membranes	#19

10/29/2012　　　　　PHY 711 Fall 2012 -- Lecture 26　　　　　2

Review of wave equation in one-dimension – here $\mu(x,t)$ can describe either a longitudinal or transverse wave.

Traveling wave solutions --

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x - ct) + g(x + ct)$$

satisfies the wave equation.

10/29/2012　　　　　PHY 711 Fall 2012 -- Lecture 26　　　　　3

Initial value problem: $\mu(x,0)=\phi(x)$ and $\dfrac{\partial\mu}{\partial t}(x,0)=\psi(x)$

then: $\mu(x,0)=\phi(x)=f(x)+g(x)$

$$\frac{\partial\mu}{\partial t}(x,0)=\psi(x)=-c\left(\frac{df(x)}{dx}-\frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x)-g(x)=-\frac{1}{c}\int^{x}\psi(x')dx'$$

For each x, find $f(x)$ and $g(x)$:

$$f(x)=\frac{1}{2}\left(\phi(x)-\frac{1}{c}\int^{x}\psi(x')dx'\right)$$

$$g(x)=\frac{1}{2}\left(\phi(x)+\frac{1}{c}\int^{x}\psi(x')dx'\right)$$

$$\Rightarrow \mu(x,t)=\frac{1}{2}\left(\phi(x-ct)+\phi(x+ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(x')dx'$$

10/29/2012   PHY 711 Fall 2012 -- Lecture 26   4

Standing wave solutions of wave equation:

$$\frac{\partial^2\mu}{\partial t^2}-c^2\frac{\partial^2\mu}{\partial x^2}=0$$

with $\mu(0,t)=\mu(L,t)=0$.

Assume: $\mu(x,t)=\Re\left(e^{-i\omega t}\rho(x)\right)$

where $\dfrac{d^2\rho(x)}{dx^2}+k^2\rho(x)=0$ $k=\dfrac{\omega}{c}$

$$\rho_\nu(x)=A\sin\left(\frac{\nu\pi x}{L}\right)$$

$$k_\nu=\frac{\nu\pi x}{L}\qquad \omega_\nu=ck_\nu$$

10/29/2012   PHY 711 Fall 2012 -- Lecture 26   5

10/29/2012   PHY 711 Fall 2012 -- Lecture 26   6
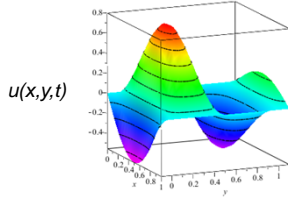
Extension of ideas to wave motion on a two-dimensional surface – elastic membrane (transverse wave; linear regime).

$u(x,y,t)$

Lagrangian density: $\mathcal{L}\left(u, \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial t}; x, y, t\right)$

$$L = \int \mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) dx\,dy$$

Hamilton's principle:

$$\delta \int_{t_1}^{t_2} L\,dt = 0$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial t)} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial x)} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial y)} = 0$$

Lagrangian density for elastic membrane:

$$\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) = \frac{1}{2}\sigma\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\tau(\nabla u)^2$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial t)} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial x)} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial y)} = 0$$

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Two-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Standing wave solutions:

$$u(x, y, t) = \Re\left(e^{-i\omega t}\rho(x, y)\right)$$

$$\left(\nabla^2 + k^2\right)\rho(x, y) = 0 \qquad \text{where } k = \frac{\omega}{c}$$

Consider a rectangular boundary:

Clamped boundary conditions :

$$\rho(0,y) = \rho(a,y) = \rho(x,0) = \rho(x,b) = 0$$

$$\Rightarrow \rho_{mn}(x,y) = A\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$k_{mn}^{\;2} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn} = ck_{mn}$$
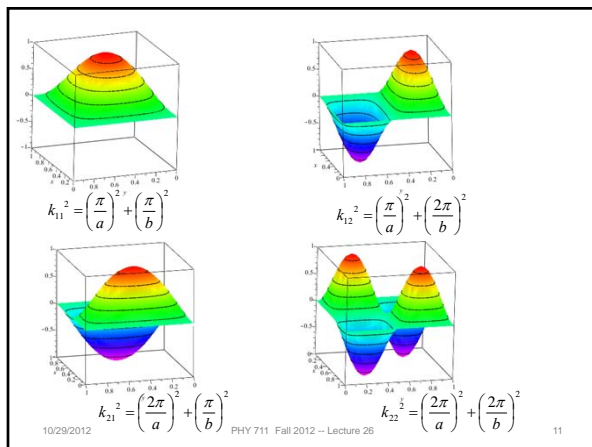
10/29/2012   PHY 711 Fall 2012 -- Lecture 26   10

$$k_{11}^{\;2} = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 \qquad\qquad k_{12}^{\;2} = \left(\frac{\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2$$

$$k_{21}^{\;2} = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 \qquad\qquad k_{22}^{\;2} = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2$$

10/29/2012   PHY 711 Fall 2012 -- Lecture 26   11

More general boundary conditions:

$$\tau\nabla u\big|_b = \kappa u\big|_b \qquad \text{represents boundard side constrained with spring}$$
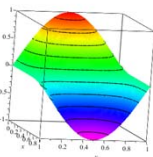
$$\tau\nabla u\big|_b = 0 \qquad \text{represents "free" side}$$

Mixed boundary conditions :

$$\rho(x,0) = \rho(x,b) = \frac{\partial\rho(0,y)}{\partial x} = \frac{\partial\rho(a,y)}{\partial x} = 0$$

$$\Rightarrow \rho_{mn}(x,y) = A\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$k_{mn}^{\;2} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn} = ck_{mn}$$

$$k_{11}^{\;2} = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$

10/29/2012   PHY 711 Fall 2012 -- Lecture 26

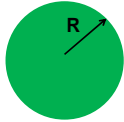Consider a circular boundary:

Clamped boundary conditions for $\rho(r,\varphi)$:
$$\rho(R,\varphi) = 0$$

$$\left(\nabla^2 + k^2\right)\rho(r,\varphi) = 0 \qquad \text{where } k = \frac{\omega}{c}$$

In cylindrical coordinate system

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

Assume: $\quad \rho(r,\varphi) = f(r)\Phi(\varphi)$

Let: $\qquad \Phi(\varphi) = e^{im\varphi}$

Note: $\qquad \Phi(\varphi) = \Phi(\varphi + 2\pi)$

$$\Rightarrow m = \text{integer}$$

10/29/2012   PHY 711 Fall 2012 -- Lecture 26   13
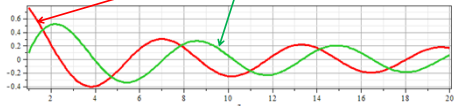

Consider circular boundary -- continued

Differential equation for radial function:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} + k^2\right)f(r) = 0$$

$\Rightarrow$ Bessel equation of integer order with transcendential solutions

    Cylindrical Bessel function $J_m(z)$

    Cylindrical Neumann function $N_m(z)$

10/29/2012   PHY 711 Fall 2012 -- Lecture 26   14

Some properties of Bessel functions

Asending series: $\quad J_m(z) = \left(\frac{z}{2}\right)^m \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+m)!}\left(\frac{z}{2}\right)^{2j}$

Recursion relations: $\quad J_{m-1}(z) + J_{m+1}(z) = \frac{2m}{z}J_m(z)$

$$J_{m-1}(z) - J_{m+1}(z) = 2\frac{dJ_m(z)}{dz}$$

Asymptotic form: $\quad J_m(z) \xrightarrow{z \gg 1} \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right)$

Zeros of Bessel functions $\quad J_m(z_{mn}) = 0$

$m = 0: \quad z_{0n} = 2.406,\ 5.520,\ 8.654,...$

$m = 1: \quad z_{1n} = 3.832,\ 7.016,\ 10.173,...$

$m = 2: \quad z_{2n} = 5.136,\ 8.417,\ 11.620,...$

10/29/2012   PHY 711 Fall 2012 -- Lecture 26   15

Some properties of Bessel functions -- continued

Note : It is possible to prove the following

identity for the functions $J_m\left(\dfrac{z_{mn}}{R}r\right)$:

$$\int_0^R J_m\left(\frac{z_{mn}}{R}r\right)J_m\left(\frac{z_{mn'}}{R}r\right)r\,dr = \frac{R^2}{2}\left(J_{m+1}(z_{mn})\right)^2 \delta_{nn'}$$

Returning to differential equation for radial function :

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

$$\Rightarrow f_{mn}(r)=AJ_m\left(\frac{z_{mn}}{R}r\right); \qquad k_{mn}=\frac{z_{mn}}{R}$$

10/29/2012   PHY 711 Fall 2012 -- Lecture 26   16

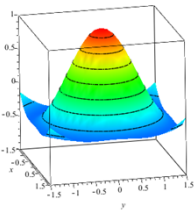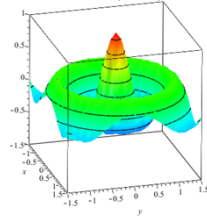$$f_{01}(r)=AJ_0\left(\frac{z_{01}}{R}r\right) \qquad f_{02}(r)=AJ_0\left(\frac{z_{02}}{R}r\right)$$

10/29/2012   PHY 711 Fall 2012 -- Lecture 26   17