PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 26:

Traveling and standing waves in 1 and 2 dimensions

1. **Brief review of 1-d from Chap. 7**
2. **Waves in 2-d elastic membranes – Chap. 8**

6	Mon, 9/10/2012	Chap. 3	Calculus of Variation	#6
7	Wed, 9/12/2012	Chap. 3	Calculus of Variation continued	
8	Fri, 9/14/2012	Chap. 3	Lagrangian	#7
9	Mon, 9/17/2012	Chap. 3 & 6	Lagrangian	#8
10	Wed, 9/19/2012	Chap. 3 & 6	Lagrangian	#9
11	Fri, 9/21/2012	Chap. 3 & 6	Lagrangian	#10
12	Mon, 9/24/2012	Chap. 3 & 6	Lagrangian and Hamiltonian	#11
13	Wed, 9/26/2012	Chap. 6	Lagrangian and Hamiltonian	#12
14	Fri, 9/28/2012	Chap. 6	Lagrangian and Hamiltonian	#13
15	Mon, 10/01/2012	Chap. 4	Small oscillations	#14
16	Wed, 10/03/2012	Chap. 4	Small oscillations	#15
17	Fri, 10/05/2012	Chap. 4	Small oscillations	
18	Mon, 10/08/2012	Chap. 7	Wave equation	Take Home Exam
19	Wed, 10/10/2012	Chap. 7	Wave equation	Take Home Exam
20	Fri, 10/12/2012	Chap. 7	Wave equation	Take Home Exam
21	Mon, 10/15/2012	Chap. 7	Wave equation	Exam due
22	Wed, 10/17/2012	Chap. 7, 5	Moment of inertia	
	Fri, 10/19/2012		Fall break	
23	Mon, 10/22/2012	Chap. 5	Rigid body rotation	#16
24	Wed, 10/24/2012	Chap. 5	Rigid body rotation	#17
25	Fri, 10/26/2012	Chap. 5	Rigid body rotation	#18
26	Mon, 10/29/2012	Chap. 8	Waves in elastic membranes	#19

Review of wave equation in one-dimension – here $\mu(x,t)$ can describe either a longitudinal or transverse wave.

Traveling wave solutions --

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x - ct) + g(x + ct)$$

satisfies the wave equation.

Initial value problem : $\mu(x,0) = \phi(x)$ and $\dfrac{\partial \mu}{\partial t}(x,0) = \psi(x)$

then : $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

For each x, find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$g(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2}\left(\phi(x-ct) + \phi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

Standing wave solutions of wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

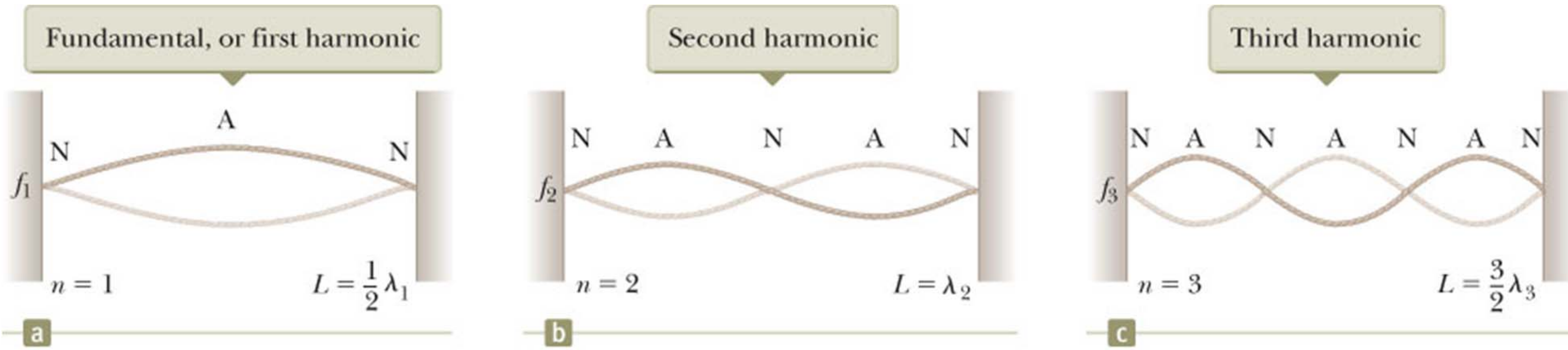
with $\mu(0,t) = \mu(L,t) = 0.$

Assume: $\mu(x,t) = \Re\left(e^{-i\omega t} \rho(x)\right)$

where $\frac{d^2 \rho(x)}{dx^2} + k^2 \rho(x) = 0$ $\qquad k = \frac{\omega}{c}$

$$\rho_\nu(x) = A \sin\left(\frac{\nu \pi x}{L}\right)$$

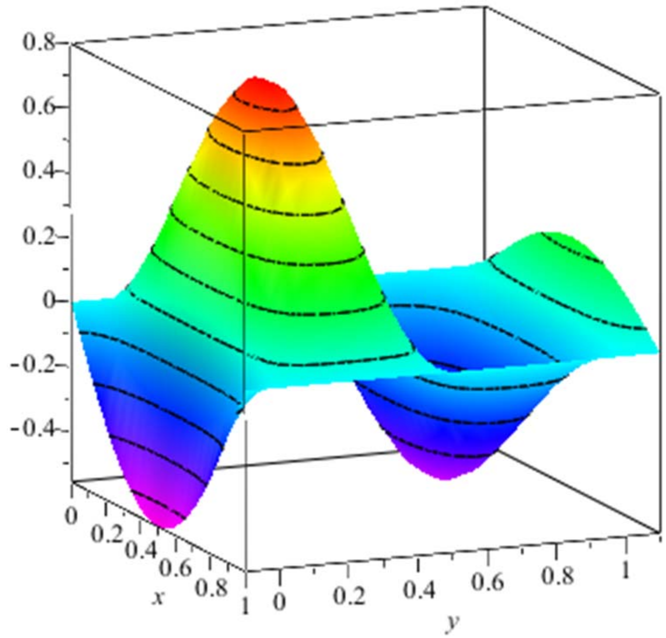
$$k_\nu = \frac{\nu \pi x}{L} \qquad \omega_\nu = c k_\nu$$



Fundamental, or first harmonic
A
N
N
f_1
$n = 1$
$L = \frac{1}{2}\lambda_1$
a
Second harmonic
N A N A N
f_2
$n = 2$
$L = \lambda_2$
b
Third harmonic
N A N A N A N
f_3
$n = 3$
$L = \frac{3}{2}\lambda_3$
c

Extension of ideas to wave motion on a two-dimensional surface – elastic membrane (transverse wave; linear regime).

u(x,y,t)



Lagrangian density: $\mathcal{L}\left(u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial t};x,y,t\right)$

$$L=\int\mathcal{L}\left(u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial t};x,y,t\right)dxdy$$

Hamilton's principle:

$$\delta\int_{t_1}^{t_2}L\,dt=0$$

$$\frac{\partial\mathcal{L}}{\partial u}-\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial u/\partial t)}-\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial(\partial u/\partial x)}-\frac{\partial}{\partial y}\frac{\partial\mathcal{L}}{\partial(\partial u/\partial y)}=0$$

Lagrangian density for elastic membrane:

$$\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) = \frac{1}{2}\sigma\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\tau(\nabla u)^2$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial t)} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial x)} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial y)} = 0$$

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Two-dimensional wave equation:

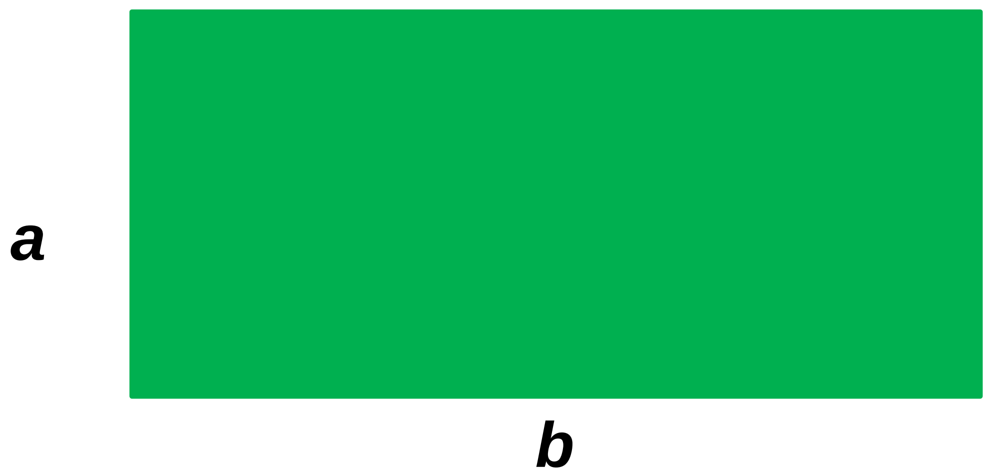
$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Standing wave solutions:

$$u(x, y, t) = \Re\left(e^{-i\omega t}\rho(x, y)\right)$$

$$\left(\nabla^2 + k^2\right)\rho(x, y) = 0 \qquad \text{where } k = \frac{\omega}{c}$$

Consider a rectangular boundary:

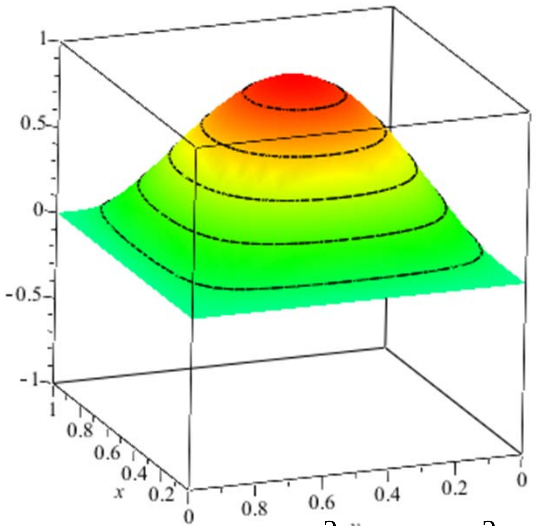


Clamped boundary conditions:

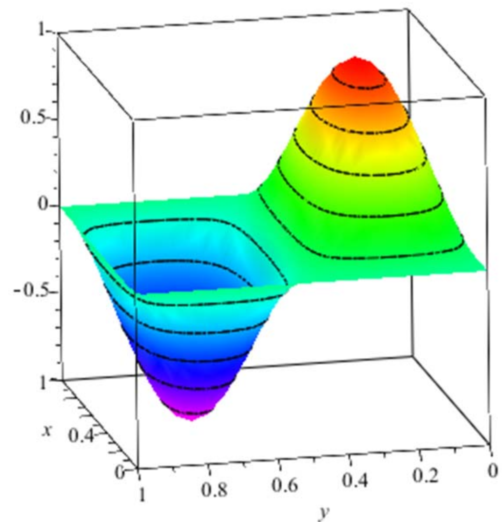
$$\rho(0,y)=\rho(a,y)=\rho(x,0)=\rho(x,b)=0$$

$$\Rightarrow \rho_{mn}(x,y)=A\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

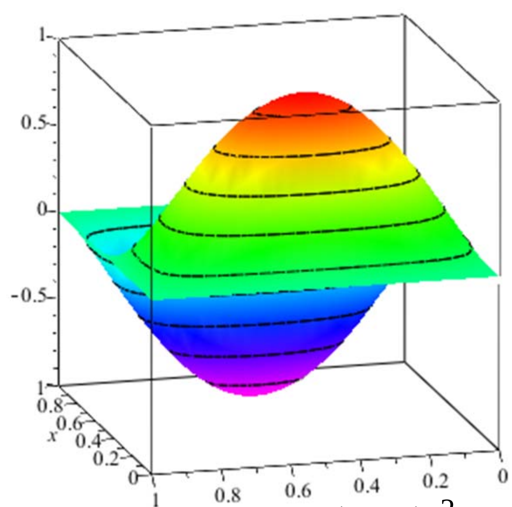
$$k_{mn}{}^2=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn}=ck_{mn}$$



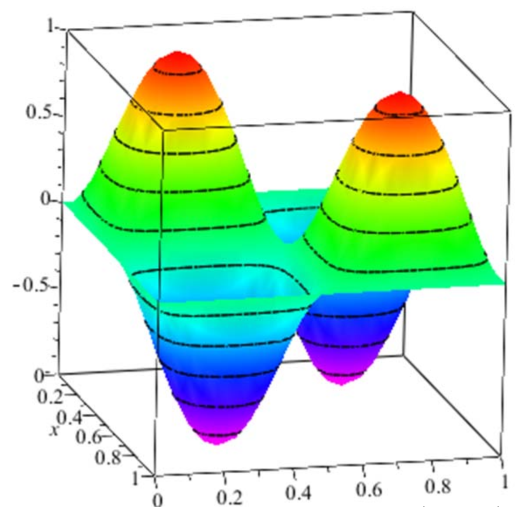
$$k_{11}{}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$



$$k_{12}{}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2$$



$$k_{21}{}^2 = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$



$$k_{22}{}^2 = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2$$

More general boundary conditions:

$$\tau \nabla u\big|_b = \kappa u\big|_b \qquad \text{represents boundard side constrained with spring}$$

$$\tau \nabla u\big|_b = 0 \qquad \text{represents "free" side}$$

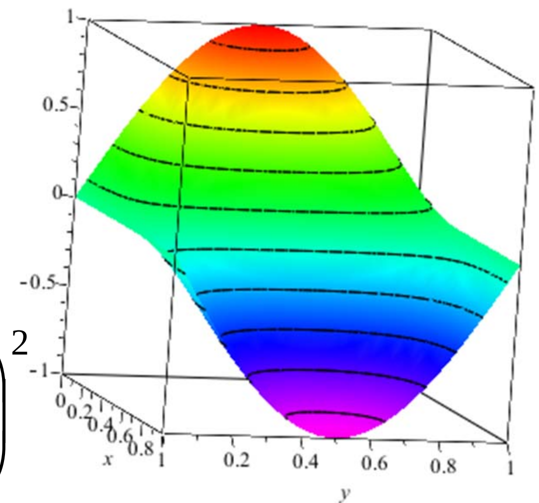
Mixed boundary conditions:

$$\rho(x,0) = \rho(x,b) = \frac{\partial \rho(0,y)}{\partial x} = \frac{\partial \rho(a,y)}{\partial x} = 0$$

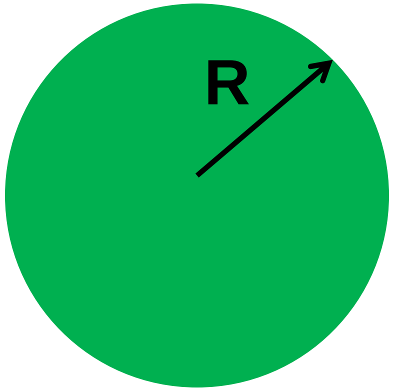
$$\Rightarrow \rho_{mn}(x,y) = A \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$k_{mn}{}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn} = c k_{mn}$$

$$k_{11}{}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$



Consider a circular boundary:



Clamped boundary conditions for $\rho(r,\varphi)$:

$$\rho(R,\varphi)=0$$

$$\left(\nabla^2+k^2\right)\rho(r,\varphi)=0 \qquad \text{where } k=\frac{\omega}{c}$$

In cylindrical coordinate system

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

Assume: $\rho(r,\varphi)=f(r)\Phi(\varphi)$

Let: $\Phi(\varphi)=e^{im\varphi}$

Note: $\Phi(\varphi)=\Phi(\varphi+2\pi)$

$$\Rightarrow m=\text{integer}$$

Consider circular boundary -- continued

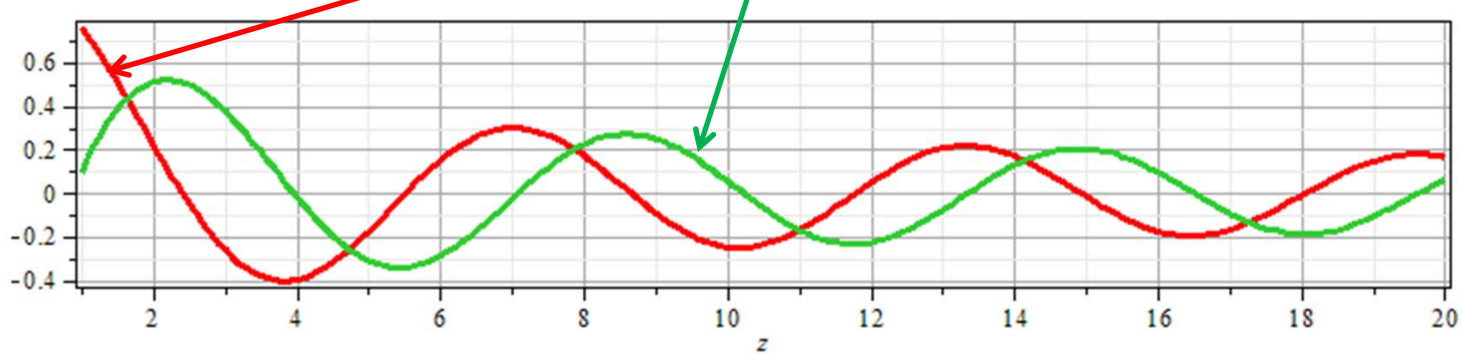
Differential equation for radial function :

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

$\Rightarrow$ Bessel equation of integer order with transcendential solutions

Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$



Some properties of Bessel functions

Asending series: $$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+m)!}\left(\frac{z}{2}\right)^{2j}$$

Recursion relations: $$J_{m-1}(z) + J_{m+1}(z) = \frac{2m}{z} J_m(z)$$

$$J_{m-1}(z) - J_{m+1}(z) = 2\frac{dJ_m(z)}{dz}$$

Asymptotic form: $$J_m(z) \xrightarrow[z>>1]{} \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

Zeros of Bessel functions $J_m(z_{mn}) = 0$

$m = 0$: $z_{0n} = 2.406,\ 5.520,\ 8.654,...$

$m = 1$: $z_{1n} = 3.832,\ 7.016,\ 10.173,...$

$m = 2$: $z_{2n} = 5.136,\ 8.417,\ 11.620,...$

Some properties of Bessel functions -- continued

Note: It is possible to prove the following identity for the functions $J_m\left(\frac{z_{mn}}{R}r\right)$:

$$\int_0^R J_m\left(\frac{z_{mn}}{R}r\right)J_m\left(\frac{z_{mn'}}{R}r\right)rdr = \frac{R^2}{2}\left(J_{m+1}\left(z_{mn}\right)\right)^2\delta_{nn'}$$

Returning to differential equation for radial function:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

$$\Rightarrow \; f_{mn}(r) = AJ_m\left(\frac{z_{mn}}{R}r\right); \qquad k_{mn} = \frac{z_{mn}}{R}$$

$$f_{01}(r) = AJ_0\left(\frac{z_{01}}{R}r\right)$$

$$f_{02}(r) = AJ_0\left(\frac{z_{02}}{R}r\right)$$