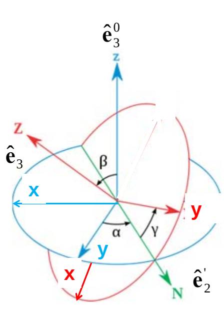
PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 25:

Rigid body rotational motion (Chap. 5)

1. **Motion of a symmetric top**

11	Fri, 9/21/2012	Chap. 3 & 6	Lagrangian	#10
12	Mon, 9/24/2012	Chap. 3 & 6	Lagrangian and Hamiltonian	#11
13	Wed, 9/26/2012	Chap. 6	Lagrangian and Hamiltonian	#12
14	Fri, 9/28/2012	Chap. 6	Lagrangian and Hamiltonian	#13
15	Mon, 10/01/2012	Chap. 4	Small oscillations	#14
16	Wed, 10/03/2012	Chap. 4	Small oscillations	#15
17	Fri, 10/05/2012	Chap. 4	Small oscillations	
18	Mon, 10/08/2012	Chap. 7	Wave equation	Take Home Exam
19	Wed, 10/10/2012	Chap. 7	Wave equation	Take Home Exam
20	Fri, 10/12/2012	Chap. 7	Wave equation	Take Home Exam
21	Mon, 10/15/2012	Chap. 7	Wave equation	Exam due
22	Wed, 10/17/2012	Chap. 7, 5	Moment of inertia	
	Fri, 10/19/2012		Fall break	
23	Mon, 10/22/2012	Chap. 5	Rigid body rotation	#16
24	Wed, 10/24/2012	Chap. 5	Rigid body rotation	#17
25	Fri, 10/26/2012	Chap. 5	Rigid body rotation	#18



$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2' + \dot{\gamma}\,\hat{\mathbf{e}}_3$$

$$\tilde{\boldsymbol{\omega}} = \left[\dot{\alpha}(-\sin\beta\cos\gamma) + \dot{\beta}\sin\gamma\right]\hat{\mathbf{e}}_1 + \left[\dot{\alpha}(\sin\beta\sin\gamma) + \dot{\beta}\cos\gamma\right]\hat{\mathbf{e}}_2 + \left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]\hat{\mathbf{e}}_3$$

Rotational kinetic energy

$$T(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \frac{1}{2}I_1\tilde{\omega}_1^2 + \frac{1}{2}I_2\tilde{\omega}_2^2 + \frac{1}{2}I_3\tilde{\omega}_3^2$$
$$= \frac{1}{2}I_1\left[\dot{\alpha}(-\sin\beta\cos\gamma) + \dot{\beta}\sin\gamma\right]^2$$
$$+ \frac{1}{2}I_2\left[\dot{\alpha}(\sin\beta\sin\gamma) + \dot{\beta}\cos\gamma\right]^2$$
$$+ \frac{1}{2}I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]^2$$

If $I_1 = I_2$:

$$T(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2\right) + \frac{1}{2}I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]^2$$

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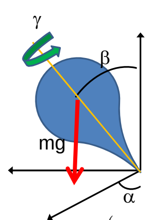
Motion of a symmetric top under the influence of the torque of gravity:



$$L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2\right) +$$
$$\frac{1}{2}I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]^2 - Mgl\cos\beta$$

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$$L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2\right) +$$
$$\frac{1}{2}I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]^2 - Mgl\cos\beta$$

Constants of the motion:

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1\dot{\alpha}\sin^2\beta + I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]\cos\beta$$
$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]$$
$$E = \frac{1}{2}I_1\dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + V_{eff}(\beta)$$
$$L(\beta,\dot{\beta}) = \frac{1}{2}I_1\dot{\beta}^2 + \frac{(p_\alpha - p_\gamma\cos\beta)^2}{2I_1\sin^2\beta} + \frac{p_\gamma^2}{2I_3} - Mgl\cos\beta$$
$$V_{eff}(\beta) = \frac{(p_\alpha - p_\gamma\cos\beta)^2}{2I_1\sin^2\beta} + Mgl\cos\beta$$

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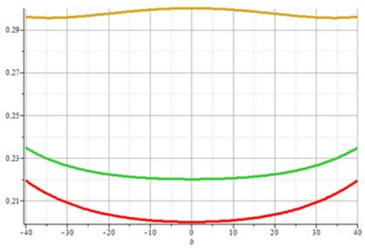
$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$

Stable/unstable solutions near β=0



Suppose $p_\alpha = p_\gamma$ and $\beta \approx 0$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + \frac{p_\gamma^2}{2I_3} + Mgl\cos\beta$$

$$E' \approx \frac{1}{2} I_1 \dot{\beta}^2 = \frac{p_\gamma{}^2}{2I_1} \frac{\left(1 - 1 + \frac{1}{2}\beta^2\right)^2}{\beta^2} + Mgl\left(1 - \tfrac{1}{2}\beta^2\right)$$

$$= \left(\frac{p_\gamma{}^2}{8I_1} - \frac{Mgl}{2}\right)\beta^2 + Mgl$$

$\Rightarrow$ Stable solution if

$$p_\gamma \geq \sqrt{4MglI_1}$$

More general case: