PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 24:

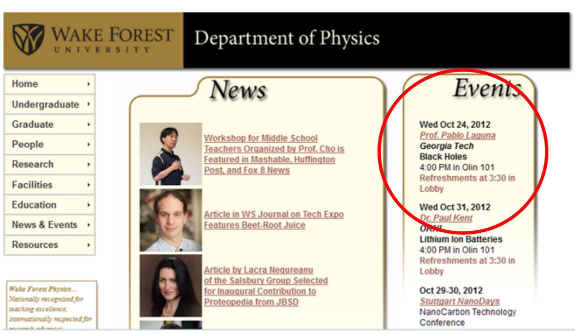
Rigid body rotational motion (Chap. 5)

1. Torque free
2. Euler angles
3. Motion of a symmetric top

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/29/2012	Chap. 1	Review of basic principles;Scattering theory	#1
2	Fri, 8/31/2012	Chap. 1	Scattering theory continued	#2
3	Mon, 9/03/2012	Chap. 1	Scattering theory continued	#3
4	Wed, 9/05/2012	Chap. 1 & 2	Scattering theory/Accelerated coordinate frame	#4
5	Fri, 9/07/2012	Chap. 2	Accelerated coordinate frame	#5
6	Mon, 9/10/2012	Chap. 3	Calculus of Variation	#6
7	Wed, 9/12/2012	Chap. 3	Calculus of Variation continued	
8	Fri, 9/14/2012	Chap. 3	Lagrangian	#7
9	Mon, 9/17/2012	Chap. 3 & 6	Lagrangian	#8
10	Wed, 9/19/2012	Chap. 3 & 6	Lagrangian	#9
11	Fri, 9/21/2012	Chap. 3 & 6	Lagrangian	#10
12	Mon, 9/24/2012	Chap. 3 & 6	Lagrangian and Hamiltonian	#11
13	Wed, 9/26/2012	Chap. 6	Lagrangian and Hamiltonian	#12
14	Fri, 9/28/2012	Chap. 6	Lagrangian and Hamiltonian	#13
15	Mon, 10/01/2012	Chap. 4	Small oscillations	#14
16	Wed, 10/03/2012	Chap. 4	Small oscillations	#15
17	Fri, 10/05/2012	Chap. 4	Small oscillations	
18	Mon, 10/08/2012	Chap. 7	Wave equation	Take Home Exam
19	Wed, 10/10/2012	Chap. 7	Wave equation	Take Home Exam
20	Fri, 10/12/2012	Chap. 7	Wave equation	Take Home Exam
21	Mon, 10/15/2012	Chap. 7	Wave equation	Exam due
22	Wed, 10/17/2012	Chap. 7, 5	Moment of inertia	
	Fri, 10/19/2012		Fall break	
23	Mon, 10/22/2012	Chap. 5	Rigid body rotation	#16
24	Wed, 10/24/2012	Chap. 5	Rigid body rotation	#17



FOREST
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Department of Physics

WFU Physics Colloquium

TITLE: Black holes and gravitational waves: The quest to show if Einstein was right

SPEAKER: Professor Pablo Laguna ,

Center for Relativistic Astrophysics
School of Physics
Georgia Institute of Technology

TIME: Wednesday October 24, 2012

PLACE: Room 101 Olin Physical Laboratory

Refreshments will be served at 3:30 PM in the Olin Lounge. All interested persons are cordially invited to attend.

Comment about Exam problem #1:

Consider a Rutherford scattering experiment analyzed in the center of mass frame of reference in which the ratio of the (repulsive) potential to the center-of-mass energy is described by a modified interaction of the form:

$$\frac{V(r)}{E} = \begin{cases} \frac{\kappa}{r} & \text{for} \quad r < r_{\max} \\ 0 & \text{for} \quad r > r_{\max}. \end{cases}$$

(a) Using the approach described in Problem 1.15 of your textbook, find an expression for $\theta(b)$, the scattering angle θ as a function of the imbact parameter b. (Simplify the expression, but you need not explicitly solve for b.)

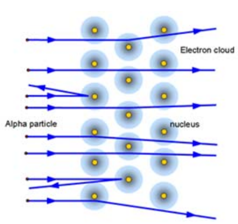
(b) Show that in the limit $r_{\max} \to \infty$, your result is consistent with the Rutherford scattering result.

(c) For the modified interaction, assume that

$$\frac{\kappa}{r_{\max}} \gg \frac{b}{r_{\max}},$$

so that terms including $(b/r_{\max})$ may be neglected. Find the form of the scattering cross section for this case and compare it with the pure Rutherford scattering cross section.

Motivation for model:



Au target particles consist of Au nuclei which are screened by electrons.

http://tap.iop.org/atoms/rutherford/img_full_47190.gif

Calculation of the scattering angle as a function of impact parameter b:

$$\theta(b) = \pi - 2b\int_{r_{\min}}^{r_{\max}} dr\,\frac{1}{r\sqrt{r^2-\kappa r-b^2}} - 2b\int_{r_{\max}}^{\infty} dr\,\frac{1}{r\sqrt{r^2-b^2}}$$

$$= \pi - 2\tan^{-1}\left(\frac{-2b^2-\kappa r}{2b\sqrt{r^2-\kappa r-b^2}}\right)\Bigg|_{r_{\min}}^{r_{\max}} - 2\tan^{-1}\left(\frac{-2b^2}{2b\sqrt{r^2-b^2}}\right)\Bigg|_{r_{\max}}^{\infty}$$

$$= 2\tan^{-1}\left(\frac{2b^2+\kappa r_{\max}}{2b\sqrt{r_{\max}{}^2-\kappa r_{\max}-b^2}}\right) + 2\tan^{-1}\left(\frac{2b^2}{2b\sqrt{r_{\max}{}^2-b^2}}\right)$$

$$= 2\tan^{-1}\left(\frac{(b/r_{\max})+(\kappa/2b)}{\sqrt{1-(\kappa/r_{\max})-(b/r_{\max})^2}}\right) + 2\tan^{-1}\left(\frac{(b/r_{\max})}{\sqrt{1-(b/r_{\max})^2}}\right)$$

$$\tan\left(\frac{\theta}{2}-\alpha\right) = \frac{(b/r_{\max})+(\kappa/2b)}{\sqrt{1-(\kappa/r_{\max})-(b/r_{\max})^2}}$$

$$\tan^2\left(\frac{\theta}{2}-\alpha\right) = \frac{(b/r_{\max})^2+(\kappa/r_{\max})+(\kappa/2b)^2}{1-(\kappa/r_{\max})-(b/r_{\max})^2}$$

where $\tan\alpha = \frac{(b/r_{\max})}{\sqrt{1-(b/r_{\max})^2}}$

Back to rotational motion

Time rate of change of angular momentum

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{L}$$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor:

$$\vec{\mathbf{I}}\cdot\hat{\mathbf{e}}_i = I_i\hat{\mathbf{e}}_i \qquad \boldsymbol{\omega} = \tilde{\omega}_1\hat{\mathbf{e}}_1+\tilde{\omega}_2\hat{\mathbf{e}}_2+\tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$\mathbf{L} = I_1\tilde{\omega}_1\hat{\mathbf{e}}_1+I_2\tilde{\omega}_2\hat{\mathbf{e}}_2+I_3\tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$\frac{d\mathbf{L}}{dt} = I_1\dot{\tilde{\omega}}_1\hat{\mathbf{e}}_1+I_2\dot{\tilde{\omega}}_2\hat{\mathbf{e}}_2+I_3\dot{\tilde{\omega}}_3\hat{\mathbf{e}}_3+\tilde{\omega}_2\tilde{\omega}_3(I_3-I_2)\hat{\mathbf{e}}_1 + \tilde{\omega}_3\tilde{\omega}_1(I_1-I_3)\hat{\mathbf{e}}_2+\tilde{\omega}_1\tilde{\omega}_2(I_2-I_1)\hat{\mathbf{e}}_3$$

Descriptions of rotation about a given origin -- continued

Note that the torque equation

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{L} = \boldsymbol{\tau}$$

is very difficult to solve directly in the body fixed frame.

For $\boldsymbol{\tau} = 0$ we can solve the Euler equations :

$$\frac{d\mathbf{L}}{dt} = I_1\dot{\tilde{\omega}}_1\hat{\mathbf{e}}_1 + I_2\dot{\tilde{\omega}}_2\hat{\mathbf{e}}_2 + I_3\dot{\tilde{\omega}}_3\hat{\mathbf{e}}_3 + \tilde{\omega}_2\tilde{\omega}_3(I_3 - I_2)\hat{\mathbf{e}}_1$$
$$+ \tilde{\omega}_3\tilde{\omega}_1(I_1 - I_3)\hat{\mathbf{e}}_2 + \tilde{\omega}_1\tilde{\omega}_2(I_2 - I_1)\hat{\mathbf{e}}_3 = 0$$

Euler equations for rotation in body fixed frame :

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3(I_3 - I_2) = 0$$
$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1(I_1 - I_3) = 0$$
$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2(I_2 - I_1) = 0$$

Solution for asymmetric top -- $I_3 \neq I_2 \neq I_1$:

Suppose : $\dot{\tilde{\omega}}_3 \approx 0$ Define : $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$

Define : $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3(I_3 - I_2) = 0$$
$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1(I_1 - I_3) = 0$$
$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2(I_2 - I_1) = 0$$

$\dot{\tilde{\omega}}_3 \approx 0$ Define : $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$ $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

$\dot{\tilde{\omega}}_1 = -\Omega_1\tilde{\omega}_2$ $\dot{\tilde{\omega}}_2 = \Omega_2\tilde{\omega}_1$

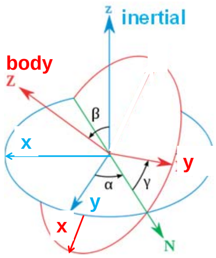
If Ω_1 and Ω_2 are both positive or both negative :

$$\tilde{\omega}_1(t) \approx A\cos\left(\sqrt{\Omega_1\Omega_2}\,t + \varphi\right)$$

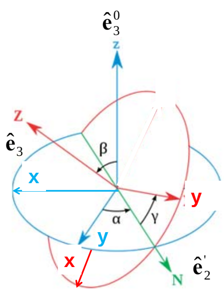
$$\tilde{\omega}_2(t) \approx A\sqrt{\frac{\Omega_2}{\Omega_1}}\sin\left(\sqrt{\Omega_1\Omega_2}\,t + \varphi\right)$$

$\Rightarrow$ If Ω_1 and Ω_2 have opposite signs, solution is unstable.

Transformation between body-fixed and inertial coordinate systems – Euler angles



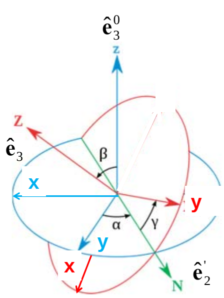
http://en.wikipedia.org/wiki/Euler_angles



$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$

Need to express all components in body-fixed frame:

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$



$$\hat{\mathbf{e}}_2^{'} = \sin\gamma\,\hat{\mathbf{e}}_1 + \cos\gamma\,\hat{\mathbf{e}}_2$$

Matrix representation :

$$\hat{\mathbf{e}}_2^{'} = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix}$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$

$$\hat{\mathbf{e}}_3^0 = -\sin\beta\,\hat{\mathbf{e}}'_1 + \cos\beta\,\hat{\mathbf{e}}'_3$$

Matrix representation :

$$\hat{\mathbf{e}}_3^0 = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix}$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix} + \dot{\beta}\begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix} + \dot{\gamma}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

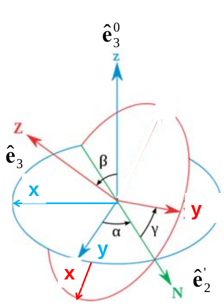
$$\tilde{\boldsymbol{\omega}} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix} + \dot{\beta}\begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix} + \dot{\gamma}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{\omega}_1 = \dot{\alpha}(-\sin\beta\cos\gamma) + \dot{\beta}\sin\gamma$$

$$\tilde{\omega}_2 = \dot{\alpha}(\sin\beta\sin\gamma) + \dot{\beta}\cos\gamma$$

$$\tilde{\omega}_3 = \dot{\alpha}\cos\beta + \dot{\gamma}$$



$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$

$$\begin{aligned} \tilde{\boldsymbol{\omega}} = &\left[\dot{\alpha}(-\sin\beta\cos\gamma) + \dot{\beta}\sin\gamma\right]\hat{\mathbf{e}}_1 \\ &+ \left[\dot{\alpha}(\sin\beta\sin\gamma) + \dot{\beta}\cos\gamma\right]\hat{\mathbf{e}}_2 \\ &+ \left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]\hat{\mathbf{e}}_3 \end{aligned}$$

Rotational kinetic energy

$$T(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \frac{1}{2}I_1\tilde{\omega}_1^2 + \frac{1}{2}I_2\tilde{\omega}_2^2 + \frac{1}{2}I_3\tilde{\omega}_3^2$$
$$= \frac{1}{2}I_1\left[\dot{\alpha}(-\sin\beta\cos\gamma) + \dot{\beta}\sin\gamma\right]^2$$
$$+ \frac{1}{2}I_2\left[\dot{\alpha}(\sin\beta\sin\gamma) + \dot{\beta}\cos\gamma\right]^2$$
$$+ \frac{1}{2}I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]^2$$

If $I_1 = I_2$:

$$T(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2\right) + \frac{1}{2}I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]^2$$