

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF Olin 103**

Plan for Lecture 18:

**Finish reading Chapter 4 and start
reading Chapter 7**

**1. Coupled motion for extended
systems; relationship to
continuum models**

2. Wave equation

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

Date	F&W Reading	Topic	Assignment
1 Wed. 8/29/2012	Chap. 1	Review of basic principles; Scattering theory	#1
2 Fri. 8/31/2012	Chap. 1	Scattering theory continued	#2
3 Mon. 9/03/2012	Chap. 1	Scattering theory continued	#3
4 Wed. 9/05/2012	Chap. 1 & 2	Scattering theory/Accelerated coordinate frame	#4
5 Fri. 9/07/2012	Chap. 2	Accelerated coordinate frame	#5
6 Mon. 9/10/2012	Chap. 3	Calculus of Variation	#6
7 Wed. 9/12/2012	Chap. 3	Calculus of Variation continued	
8 Fri. 9/14/2012	Chap. 3	Lagrangian	#7
9 Mon. 9/17/2012	Chap. 3 & 6	Lagrangian	#8
10 Wed. 9/19/2012	Chap. 3 & 6	Lagrangian	#9
11 Fri. 9/21/2012	Chap. 3 & 6	Lagrangian	#10
12 Mon. 9/24/2012	Chap. 3 & 6	Lagrangian and Hamiltonian	#11
13 Wed. 9/26/2012	Chap. 6	Lagrangian and Hamiltonian	#12
14 Fri. 9/28/2012	Chap. 6	Lagrangian and Hamiltonian	#13
15 Mon. 10/01/2012	Chap. 4	Small oscillations	#14
16 Wed. 10/03/2012	Chap. 4	Small oscillations	#15
17 Fri. 10/05/2012	Chap. 4	Small oscillations	
18 Mon. 10/08/2012	Chap. 7	Wave equation	Take Home Exam
19 Wed. 10/10/2012	Chap. 7	Wave equation	Take Home Exam
20 Fri. 10/12/2012	Chap. 7	Wave equation	Take Home Exam
21 Mon. 10/15/2012	Chap. 7	Wave equation	Exam due

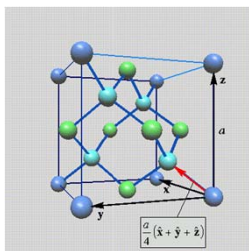
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Lattice vibrations for 3-dimensional lattice

Example: diamond lattice



Ref: http://phycomp.technion.ac.il/~nika/diamond_structure.html

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$$I(\dot{u}_j^a, \dot{u}_j^a) = \frac{1}{2} \sum_{a,j} m_a (\dot{u}_j^a)^2 - U_0 - \frac{1}{2} \sum_{a,b,j,k} u_j^a D_{jk}^{ab} u_k^b$$

Equations of motion:

$$m_a \ddot{u}_j^a = - \sum_{b,k} D_{jk}^{ab} u_k^b$$

Solution form:

$$u_j^a(t) = \frac{1}{\sqrt{m_a}} A_j^a e^{-i\omega t + i\mathbf{q} \cdot \mathbf{R}_j^a}$$

Details: $\mathbf{R}_0^a = \boldsymbol{\tau}^a + \mathbf{T}$ where $\boldsymbol{\tau}^a$ denotes unique sites and \mathbf{T} denotes replicas

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Define:

$$W_{jk}^{ab}(\mathbf{q}) = \sum_{\mathbf{T}} \frac{D_{jk}^{ab} e^{i\mathbf{q} \cdot (\boldsymbol{\tau}^a - \boldsymbol{\tau}^b)}}{\sqrt{m_a m_b}} e^{i\mathbf{q} \cdot \mathbf{T}}$$

Eigenvalue equations:

$$\omega^2 A_j^a = \sum_{b,k} W(\mathbf{q})_{jk}^{ab} A_k^b$$

In this equation the summation is only over unique atomic sites.
 => Find "dispersion curves" $\omega(\mathbf{q})$

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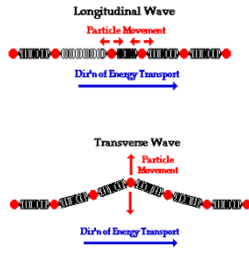
B. P. Pandy and B. Dayal, J. Phys. C. Solid State Phys. 6 2943 (1973)

Note: Longitudinal, transverse, and combination modes occur.

Figure 2. Phonon dispersion curves of diamond. Experimental points et al (1965, 1967). Δ and \circ represent the longitudinal and transverse modes

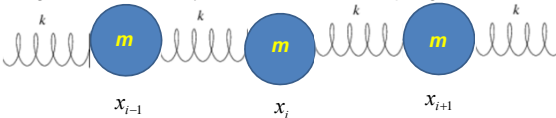
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Longitudinal versus transverse vibrations
 Images from web page:
<http://www.physicsclassroom.com/class/waves/u10l1c.cfm>



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Longitudinal case: a system of masses and springs:



$$L = T - V = \frac{1}{2} m \sum_{i=0}^{\infty} \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^{\infty} (x_{i+1} - x_i)^2$$

$$\Rightarrow m \ddot{x}_i = k(x_{i+1} - 2x_i + x_{i-1})$$

Now imagine the continuum version of this system:

$$x_i(t) \Rightarrow \mu(x_i, t) \quad \ddot{x}_i \Rightarrow \frac{\partial^2 \mu}{\partial t^2}$$


$$x_{i+1} - 2x_i + x_{i-1} \Rightarrow \frac{\partial^2 \mu}{\partial x^2} (\Delta x)^2$$

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Discrete equation: $m \ddot{x}_i = k(x_{i+1} - 2x_i + x_{i-1})$

Continuum equation: $m \frac{\partial^2 \mu}{\partial t^2} = k(\Delta x)^2 \frac{\partial^2 \mu}{\partial x^2}$

$$\frac{\partial^2 \mu}{\partial t^2} = \left(\frac{k \Delta x}{m / \Delta x} \right) \frac{\partial^2 \mu}{\partial x^2}$$

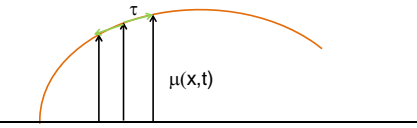

 system parameter with units of (velocity)²

For transverse oscillations on a string
 with tension τ and mass/length σ :

$$\left(\frac{k \Delta x}{m / \Delta x} \right) \Rightarrow \frac{\tau}{\sigma}$$

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Transverse displacement:



Wave equation :

$$\frac{\partial^2 \mu}{\partial t^2} = c^2 \frac{\partial^2 \mu}{\partial x^2}$$

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Lagrangian for continuous system :

Denote the generalized displacement by $\mu(x,t)$:

$$L = L\left(\mu, \frac{\partial \mu}{\partial x}, \frac{\partial \mu}{\partial t}; x, t\right)$$

Hamilton's principle :

$$\delta \int_{t_i}^{t_f} \int_{x_i}^{x_f} dx L\left(\mu, \frac{\partial \mu}{\partial x}, \frac{\partial \mu}{\partial t}; x, t\right) = 0$$

$$\Rightarrow \frac{\partial L}{\partial \mu} - \frac{\partial}{\partial x} \frac{\partial L}{\partial (\partial \mu / \partial x)} - \frac{\partial}{\partial t} \frac{\partial L}{\partial (\partial \mu / \partial t)} = 0$$

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Euler - Lagrange equations for continuous system :

$$\frac{\partial L}{\partial \mu} - \frac{\partial}{\partial x} \frac{\partial L}{\partial (\partial \mu / \partial x)} - \frac{\partial}{\partial t} \frac{\partial L}{\partial (\partial \mu / \partial t)} = 0$$

Example :

$$L = \frac{\sigma}{2} \left(\frac{\partial \mu}{\partial t}\right)^2 - \frac{\tau}{2} \left(\frac{\partial \mu}{\partial x}\right)^2$$

$$\Rightarrow \sigma \frac{\partial^2 \mu}{\partial t^2} - \tau \frac{\partial^2 \mu}{\partial x^2} = 0$$

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{for } c^2 = \frac{\tau}{\sigma}$$

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General solutions $\mu(x,t)$ to the wave equation :

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

satisfies the wave equation.

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Initial value solutions $\mu(x,t)$ to the wave equation;
attributed to D'Alembert :

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = \phi(x) \text{ and } \frac{\partial \mu}{\partial t}(x,0) = \psi(x)$$

Assume :

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

$$\text{then : } \mu(x,0) = \phi(x) = f(x) + g(x)$$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c \left(\frac{df(x)}{dx} - \frac{dg(x)}{dx} \right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c} \int \psi(x') dx'$$

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Solution -- continued : $\mu(x,t) = f(x-ct) + g(x+ct)$

$$\text{then : } \mu(x,0) = \phi(x) = f(x) + g(x)$$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c \left(\frac{df(x)}{dx} - \frac{dg(x)}{dx} \right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c} \int \psi(x') dx'$$

For each x , find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2} \left(\phi(x) - \frac{1}{c} \int \psi(x') dx' \right)$$

$$g(x) = \frac{1}{2} \left(\phi(x) + \frac{1}{c} \int \psi(x') dx' \right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2} (\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(x') dx'$$

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Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = e^{-x^2/\sigma^2} \text{ and } \frac{\partial \mu}{\partial t}(x,0) = 0$$

$$\Rightarrow \mu(x,t) = \frac{1}{2} \left(e^{-(x+ct)^2/\sigma^2} + e^{-(x-ct)^2/\sigma^2} \right)$$

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Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = 0 \text{ and } \frac{\partial \mu}{\partial t}(x,0) = -\frac{2x}{\sigma^2} e^{-x^2/\sigma^2}$$

$$\Rightarrow \mu(x,t) = \frac{1}{2c} \left(e^{-(x+ct)^2/\sigma^2} - e^{-(x-ct)^2/\sigma^2} \right)$$

Note that $\frac{\partial \mu(x,t)}{\partial t} = -\frac{1}{\sigma^2} \left((x+ct)e^{-(x+ct)^2/\sigma^2} + (x-ct)e^{-(x-ct)^2/\sigma^2} \right)$

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