

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF Olin 103**

Plan for Lecture 15:

Start reading Chapter 4

- 1. Small oscillations about equilibrium**
- 2. Normal modes**

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

Date	F&W Reading	Topic	Assignment
1 Wed, 8/29/2012	Chap. 1	Review of basic principles; Scattering theory	#1
2 Fri, 8/31/2012	Chap. 1	Scattering theory continued	#2
3 Mon, 9/03/2012	Chap. 1	Scattering theory continued	#3
4 Wed, 9/05/2012	Chap. 1 & 2	Scattering theory/Accelerated coordinate frame	#4
5 Fri, 9/07/2012	Chap. 2	Accelerated coordinate frame	#5
6 Mon, 9/10/2012	Chap. 3	Calculus of Variation	#6
7 Wed, 9/12/2012	Chap. 3	Calculus of Variation continued	#7
8 Fri, 9/14/2012	Chap. 3	Lagrangian	#8
9 Mon, 9/17/2012	Chap. 3 & 6	Lagrangian	#9
10 Wed, 9/19/2012	Chap. 3 & 6	Lagrangian	#10
11 Fri, 9/21/2012	Chap. 3 & 6	Lagrangian	#11
12 Mon, 9/24/2012	Chap. 3 & 6	Lagrangian and Hamiltonian	#12
13 Wed, 9/26/2012	Chap. 6	Lagrangian and Hamiltonian	#13
14 Fri, 9/28/2012	Chap. 6	Lagrangian and Hamiltonian	#14
15 Mon, 10/01/2012	Chap. 4	Small oscillations	#15



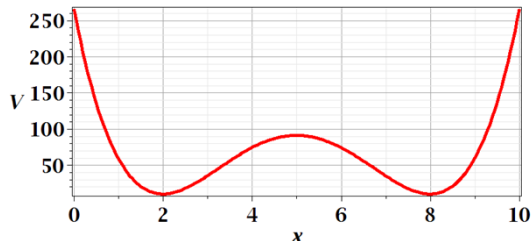
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Motivation for studying small oscillations – many interacting systems have stable and meta-stable configurations which are well approximated by:

$$V(x) \approx V(x_{eq}) + \frac{1}{2}(x - x_{eq})^2 \left. \frac{d^2V}{dx^2} \right|_{x_{eq}} = V(x_{eq}) + \frac{1}{2}k(x - x_{eq})^2$$



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Equations of motion for a single oscillator :

$$L(x, \dot{x}, t) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \quad \Rightarrow \quad m \ddot{x} = -m \omega^2 x$$

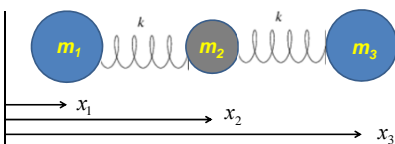
$$x(t) = A \sin(\omega t + \phi)$$

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Example – linear molecule



$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

$$- \frac{1}{2} k (x_2 - x_1 - \ell_{12})^2 - \frac{1}{2} k (x_3 - x_2 - \ell_{23})^2$$

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$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

$$- \frac{1}{2} k (x_2 - x_1 - \ell_{12})^2 - \frac{1}{2} k (x_3 - x_2 - \ell_{23})^2$$

Let: $x_1 \rightarrow x_1 - x_1^0$ $x_2 \rightarrow x_2 - x_1^0 - \ell_{12}$ $x_3 \rightarrow x_3 - x_1^0 - \ell_{12} - \ell_{23}$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2 - \frac{1}{2} k (x_2 - x_1)^2 - \frac{1}{2} k (x_3 - x_2)^2$$

Coupled equations of motion :

$$m_1 \ddot{x}_1 = k(x_2 - x_1)$$

$$m_2 \ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2) = k(x_1 - 2x_2 + x_3)$$

$$m_3 \ddot{x}_3 = -k(x_3 - x_2)$$

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Coupled equations of motion :

$$m_1 \ddot{x}_1 = k(x_2 - x_1)$$

$$m_2 \ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2) = k(x_1 - 2x_2 + x_3)$$

$$m_3 \ddot{x}_3 = -k(x_3 - x_2)$$

Let $x_i(t) = X_i^\alpha e^{-i\omega_\alpha t}$

$$-\omega_\alpha^2 m_1 X_1 = k(X_2^\alpha - X_1^\alpha)$$

$$-\omega_\alpha^2 m_2 X_2 = k(X_1^\alpha - 2X_2^\alpha + X_3^\alpha)$$

$$-\omega_\alpha^2 m_3 X_3 = -k(X_3^\alpha - X_2^\alpha)$$

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Coupled linear equations:

$$-\omega_\alpha^2 m_1 X_1^\alpha = k(X_2^\alpha - X_1^\alpha)$$

$$-\omega_\alpha^2 m_2 X_2^\alpha = k(X_1^\alpha - 2X_2^\alpha + X_3^\alpha)$$

$$-\omega_\alpha^2 m_3 X_3^\alpha = -k(X_3^\alpha - X_2^\alpha)$$

Matrix form:

$$\begin{pmatrix} k - \omega_\alpha^2 m_1 & -k & 0 \\ -k & 2k - \omega_\alpha^2 m_2 & -k \\ 0 & -k & k - \omega_\alpha^2 m_3 \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = 0$$

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Matrix form:

$$\begin{pmatrix} k - \omega_\alpha^2 m_1 & -k & 0 \\ -k & 2k - \omega_\alpha^2 m_2 & -k \\ 0 & -k & k - \omega_\alpha^2 m_3 \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = 0$$

More convenient form:

Let $Y_i \equiv \sqrt{m_i} X_i$ Equations for Y_i take the form:

$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

where $\kappa_{ij} = \kappa_{ji} \equiv \frac{k}{\sqrt{m_i m_j}}$

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Digression:

Eigenvalue properties of matrices $\mathbf{M}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$

Hermitian matrix : $H_{ij} = H_{ji}^*$

Theorem for Hermitian matrices :

$$\lambda_\alpha \text{ have real values and } \mathbf{y}_\alpha^* \cdot \mathbf{y}_\beta = \delta_{\alpha\beta}$$

Unitary matrix : $\mathbf{U}\mathbf{U}^* = \mathbf{I}$

$$|\lambda_\alpha| = 1 \text{ and } \mathbf{y}_\alpha^* \cdot \mathbf{y}_\beta = \delta_{\alpha\beta}$$

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In our case :

$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

for $m_1 = m_3 \equiv m_o$ and $m_2 \equiv m_c$ (CO_2)

$$\begin{pmatrix} \kappa_{oo} & -\kappa_{oc} & 0 \\ -\kappa_{oc} & 2\kappa_{cc} & -\kappa_{oc} \\ 0 & -\kappa_{oc} & \kappa_{oo} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

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Eigenvalues and eigenvectors :

$$\omega_1^2 = 0 \quad \begin{pmatrix} Y_1^1 \\ Y_2^1 \\ Y_3^1 \end{pmatrix} = N_1 \begin{pmatrix} \sqrt{\frac{m_o}{m_c}} \\ 1 \\ \sqrt{\frac{m_o}{m_c}} \end{pmatrix}, \quad \begin{pmatrix} X_1^1 \\ X_2^1 \\ X_3^1 \end{pmatrix} = N'_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

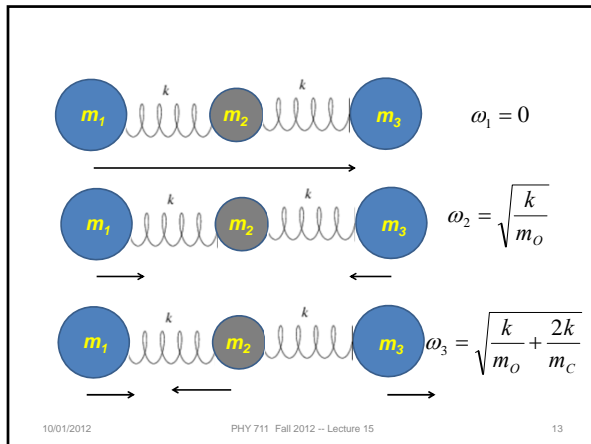
$$\omega_2^2 = \frac{k}{m_o} \quad \begin{pmatrix} Y_1^2 \\ Y_2^2 \\ Y_3^2 \end{pmatrix} = N_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} X_1^2 \\ X_2^2 \\ X_3^2 \end{pmatrix} = N'_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\omega_3^2 = \frac{k}{m_o} + \frac{2k}{m_c} \quad \begin{pmatrix} Y_1^3 \\ Y_2^3 \\ Y_3^3 \end{pmatrix} = N_3 \begin{pmatrix} 1 \\ -2\sqrt{\frac{m_o}{m_c}} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} X_1^3 \\ X_2^3 \\ X_3^3 \end{pmatrix} = N'_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

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General solution :

$$x_i(t) = \Re \left(\sum_{\alpha} C^{\alpha} X_i^{\alpha} e^{-i\omega_{\alpha} t} \right)$$

For example, normal mode amplitudes C^{α} can be determined from initial conditions

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