PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 12:

Continue reading Chapter 3 & 6

1. **Constructing the Hamiltonian**
2. **Hamilton's canonical equation**
3. **Examples**

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/29/2012	Chap. 1	Review of basic principles;Scattering theory	#1
2	Fri, 8/31/2012	Chap. 1	Scattering theory continued	#2
3	Mon, 9/03/2012	Chap. 1	Scattering theory continued	#3
4	Wed, 9/05/2012	Chap. 1 & 2	Scattering theory/Accelerated coordinate frame	#4
5	Fri, 9/07/2012	Chap. 2	Accelerated coordinate frame	#5
6	Mon, 9/10/2012	Chap. 3	Calculus of Variation	#6
7	Wed, 9/12/2012	Chap. 3	Calculus of Variation continued	
8	Fri, 9/14/2012	Chap. 3	Lagrangian	#7
9	Mon, 9/17/2012	Chap. 3 & 6	Lagrangian	#8
10	Wed, 9/19/2012	Chap. 3 & 6	Lagrangian	#9
11	Fri, 9/21/2012	Chap. 3 & 6	Lagrangian	#10
12	Mon, 9/24/2012	Chap. 3 & 6	Lagrangian and Hamiltonian	#11

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

Hamiltonian picture – continued

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

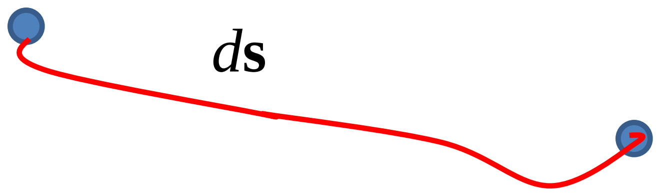
$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \qquad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \qquad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

Direct application of Hamiltonian's principle using the Hamiltonian function --



Generalized coordinates:

$$q_\sigma(\{x_i\})$$

Define -- Lagrangian: $L \equiv T - U$

$$L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)$$

$$\Rightarrow \text{Minimization integral:} \quad S = \int_{t_i}^{t_f} L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)\, dt$$

Expressed in terms of Hamiltonian:

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \Rightarrow L = \sum_\sigma \dot{q}_\sigma p_\sigma - H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

Hamilton's principle continued:

Minimization integral:

$$S=\int_{t_i}^{t_f}\left(\sum_\sigma \dot{q}_\sigma p_\sigma - H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)\right)dt$$

$$\delta S=\int_{t_i}^{t_f}\left(\sum_\sigma\left(\dot{q}_\sigma\delta p_\sigma+\delta\dot{q}_\sigma p_\sigma-\frac{\partial H}{\partial q_\sigma}\delta q_\sigma-\frac{\partial H}{\partial p_\sigma}\delta p_\sigma\right)\right)dt=0$$

$$\Rightarrow \dot{q}_\sigma=\frac{\partial H}{\partial p_\sigma}$$

$$\Rightarrow \dot{p}_\sigma=-\frac{\partial H}{\partial q_\sigma}$$

Canonical equations

Detail:

0

$$\int_{t_i}^{t_f}\left(\sum_\sigma\left(\delta\dot{q}_\sigma p_\sigma\right)\right)dt=\int_{t_i}^{t_f}\left(\sum_\sigma\left(\frac{d\left(\delta q_\sigma p_\sigma\right)}{dt}-\delta q_\sigma \dot{p}_\sigma\right)\right)dt=\sum_\sigma \delta q_\sigma p_\sigma\Big|_{t_i}^{t_f}-\int_{t_i}^{t_f}\left(\sum_\sigma\left(\delta q_\sigma \dot{p}_\sigma\right)\right)dt$$

Constants of the motion in Hamiltonian formalism

$$H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \Rightarrow \text{constant } \; q_\sigma \; \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \qquad \Rightarrow \text{constant } \; p_\sigma \; \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

$$\Rightarrow \text{constant } \; H \; \text{ if } \; \frac{\partial H}{\partial t} = 0$$

Recipe for constructing the Hamiltonian

1. Construct Lagrangian function: $L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$

2. Compute generalized momenta: $p_\sigma \equiv \dfrac{\partial L}{\partial \dot{q}_\sigma}$

3. Construct Hamiltonian expression: $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$

4. Form Hamiltonian function: $H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$

Example 1: one-dimensional potential:

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)$$

$$p_x = m\dot{x} \qquad p_y = m\dot{y} \qquad p_z = m\dot{z}$$

$$H = m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2 - \left(\tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)\right)$$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + V(z)$$

Constants: p_x, p_y, H

Example 2: Motion in a central potential

$$L = \tfrac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) - V(r)$$

$$p_r = m\dot{r} \qquad\qquad p_\phi = mr^2\dot{\phi}$$

$$H = m\dot{r}^2 + mr^2\dot{\phi}^2 - \left(\tfrac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) - V(r) \right)$$

$$= \tfrac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) + V(r)$$

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r)$$

Constants: p_φ, H

Other examples

Lagrangian for symmetric top with Euler angles α, β, γ :

$$L = L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \tfrac{1}{2} I_1\left(\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2\right) + \tfrac{1}{2} I_3\left(\dot{\alpha} \cos \beta + \dot{\gamma}\right)^2 - Mgh \cos \beta$$

$$p_\alpha = I_1 \dot{\alpha} \sin^2 \beta + I_3\left(\dot{\alpha} \cos \beta + \dot{\gamma}\right) \cos \beta$$

$$p_\beta = I_1 \dot{\beta}$$

$$p_\gamma = I_3\left(\dot{\alpha} \cos \beta + \dot{\gamma}\right)$$

$$H = \tfrac{1}{2} I_1\left(\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2\right) + \tfrac{1}{2} I_3\left(\dot{\alpha} \cos \beta + \dot{\gamma}\right)^2 + Mgh \cos \beta$$

$$H = \frac{\left(p_\alpha - p_\gamma \cos \beta\right)^2}{2 I_1 \sin^2 \beta} + \frac{p_\beta^2}{2 I_1} + \frac{p_\gamma^2}{2 I_3} + Mgh \cos \beta$$

Constants: p_α, p_γ, H

Other examples

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(\dot{x}y - \dot{y}x\right)$$

$$p_x = m\dot{x} + \frac{q}{2c} B_0 y$$

$$p_y = m\dot{y} - \frac{q}{2c} B_0 x$$

$$p_z = m\dot{z}$$

$$H = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$H = \left(\frac{p_x - \frac{q}{2c} B_0 y}{2m}\right)^2 + \left(\frac{p_y + \frac{q}{2c} B_0 x}{2m}\right)^2 + \frac{p_z^2}{2m}$$

$$\text{Constants:} \qquad p_z, H$$

Poisson brackets:

Recall:

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \Rightarrow \text{constant } \; q_\sigma \; \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \qquad \Rightarrow \text{constant } \; p_\sigma \; \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

Similarly for an arbitrary function: $F = F\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$

$$\frac{dF}{dt} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial F}{\partial t} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\frac{\partial H}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial H}{\partial q_\sigma} \right) + \frac{\partial F}{\partial t}$$

Poisson brackets -- continued:

For an arbitrary function: $F = F\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$

$$\frac{dF}{dt} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial F}{\partial t} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\frac{\partial H}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial H}{\partial q_\sigma} \right) + \frac{\partial F}{\partial t}$$

Define:

$$\left[F,G\right]_{PB} \equiv \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial G}{\partial q_\sigma} \right) = -\left[G,F\right]_{PB}$$

So that: $$\frac{dF}{dt} = \left[F,H\right]_{PB} + \frac{\partial F}{\partial t}$$

Poisson brackets -- continued:

$$[F,G]_{PB} \equiv \sum_{\sigma}\left(\frac{\partial F}{\partial q_\sigma}\frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial G}{\partial q_\sigma}\right) = -[G,F]_{PB}$$

Examples:

$$[x,x]_{PB} = 0 \qquad [x,p_x]_{PB} = 1 \qquad [x,p_y]_{PB} = 0$$

Liouville theorem

Let $D \equiv$ density of particles in phase space:

$$\frac{dD}{dt} = [D,H]_{PB} + \frac{\partial D}{\partial t} = 0$$