

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF Olin 103**

Plan for Lecture 11:
Continue reading Chapter 3 & 6
1. Constants of the motion
2. Conserved quantities

PHY 711 Classical Mechanics and Mathematical Methods

MWF 10 AM-10:50 PM OPL 103 <http://www.wfu.edu/~natalie/12phy711/>

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

Date	F&W Reading	Topic	Assignment
1 Wed, 8/29/2012	Chap. 1	Review of basic principles; Scattering theory	#1
2 Fri, 8/31/2012	Chap. 1	Scattering theory continued	#2
3 Mon, 9/03/2012	Chap. 1	Scattering theory continued	#3
4 Wed, 9/05/2012	Chap. 1 & 2	Scattering theory; Accelerated coordinate frame	#4
5 Fri, 9/07/2012	Chap. 2	Accelerated coordinate frame	#5
6 Mon, 9/10/2012	Chap. 3	Calculus of Variation	#6
7 Wed, 9/12/2012	Chap. 3	Calculus of Variation continued	
8 Fri, 9/14/2012	Chap. 3	Lagrangian	#7
9 Mon, 9/17/2012	Chap. 3 & 6	Lagrangian	#8
10 Wed, 9/19/2012	Chap. 3 & 6	Lagrangian	#9
11 Fri, 9/21/2012	Chap. 3 & 6	Lagrangian	#10

Summary of Lagrangian formalism (without constraints)

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Note that if $\frac{\partial L}{\partial q_\sigma} = 0$, then $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} = 0$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_\sigma} = (\text{constant})$$

Examples of constants of the motion:

Example 1: one-dimensional potential:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z)$$

$$\Rightarrow \frac{d}{dt}m\dot{x} = 0 \quad \Rightarrow m\dot{x} \equiv p_x \text{ (constant)}$$

$$\Rightarrow \frac{d}{dt}m\dot{y} = 0 \quad \Rightarrow m\dot{y} \equiv p_y \text{ (constant)}$$

$$\Rightarrow \frac{d}{dt}m\dot{z} = -\frac{\partial V}{\partial z}$$

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Examples of constants of the motion:

Example 2: Motion in a central potential

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$$

$$\Rightarrow \frac{d}{dt}mr^2\dot{\phi} = 0 \quad \Rightarrow mr^2\dot{\phi} \equiv p_\phi \text{ (constant)}$$

$$\Rightarrow \frac{d}{dt}m\dot{r} = mr\dot{\phi}^2 - \frac{\partial V}{\partial r} = \frac{p_\phi^2}{mr^3} - \frac{\partial V}{\partial r}$$

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Recall alternative form of Euler-Lagrange equations:

Starting from:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$$\text{Also note that: } \frac{dL}{dt} = \sum_\sigma \frac{\partial L}{\partial q_\sigma} \dot{q}_\sigma + \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \ddot{q}_\sigma + \frac{\partial L}{\partial t}$$

$$= \frac{d}{dt} \left(\sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma \right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt} \left(L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma \right) = \frac{\partial L}{\partial t}$$

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Additional constant of the motion:

$$\text{If } \frac{\partial L}{\partial t} = 0;$$

$$\text{then: } \frac{d}{dt} \left(L - \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} \right) = \frac{\partial L}{\partial t} = 0$$

$$\Rightarrow L - \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} = -E \quad (\text{constant})$$

Example 1: one-dimensional potential:

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z)$$

$$\Rightarrow \frac{d}{dt} \left(\frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z) - m\dot{x}^2 - m\dot{y}^2 - m\dot{z}^2 \right) = 0$$

$$\Rightarrow -\left(\frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(z) \right) = -E \quad (\text{constant})$$

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Additional constant of the motion -- continued:

$$\text{If } \frac{\partial L}{\partial t} = 0;$$

$$\text{then: } \frac{d}{dt} \left(L - \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} \right) = \frac{\partial L}{\partial t} = 0$$

$$\Rightarrow L - \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} = -E \quad (\text{constant})$$

Example 2: Motion in a central potential

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - V(r)$$

$$\Rightarrow \frac{d}{dt} \left(\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - V(r) - m\dot{r}^2 - mr^2 \dot{\phi}^2 \right) = 0$$

$$\Rightarrow -\left(\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + V(r) \right) = -E \quad (\text{constant})$$

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Other examples

Lagrangian for symmetric top with Euler angles α, β, γ :

$$L = L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 (\dot{\alpha} \cos \beta + \dot{\gamma})^2 - Mgh \cos \beta$$

Constants of the motion:

$$\frac{\partial L}{\partial \gamma} = 0 \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{\gamma}} = 0 \quad I_3 (\dot{\alpha} \cos \beta + \dot{\gamma}) = p_{\gamma} \quad (\text{constant})$$

$$\frac{\partial L}{\partial \alpha} = 0 \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{\alpha}} = 0 \quad I_1 \dot{\alpha} \sin^2 \beta + I_3 (\dot{\alpha} \cos \beta + \dot{\gamma}) \cos \beta = p_{\alpha} \quad (\text{constant})$$

$$E = \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} - L$$

$$= \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 (\dot{\alpha} \cos \beta + \dot{\gamma})^2 + Mgh \cos \beta$$

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Other examples

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c}B_0(\dot{y}x - \dot{x}y)$$

$$\frac{\partial L}{\partial z} = 0 \quad \Rightarrow \quad m\dot{z} = p_z \quad (\text{constant})$$

$$E = \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} - L$$

$$= m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c}B_0(\dot{y}x - \dot{x}y)$$

$$- \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{q}{2c}B_0(\dot{y}x - \dot{x}y)$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

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Lagrangian picture

For independent generalized coordinates $q_{\sigma}(t)$:

$$L = L(\{q_{\sigma}(t)\}, \{\dot{q}_{\sigma}(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_{\sigma}} - \frac{\partial L}{\partial q_{\sigma}} = 0$$

\Rightarrow Second order differential equations for $q_{\sigma}(t)$

Switching variables – Legendre transformation

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Mathematical transformations for continuous functions of several variables & Legendre transforms:

$$z(x, y) \Leftrightarrow x(y, z) ???$$

$$z(x, y) \Rightarrow dz = \left(\frac{\partial z}{\partial x} \right)_y dx + \left(\frac{\partial z}{\partial y} \right)_x dy$$

$$x(y, z) \Rightarrow dx = \left(\frac{\partial x}{\partial y} \right)_z dy + \left(\frac{\partial x}{\partial z} \right)_y dz$$

$$\text{But: } \left(\frac{\partial x}{\partial y} \right)_z = - \frac{(\partial z / \partial y)_x}{(\partial z / \partial x)_y}$$

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Mathematical transformations for continuous functions of several variables & Legendre transforms continued:

$$z(x, y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

Let $u \equiv \left(\frac{\partial z}{\partial x}\right)_y$ and $v \equiv \left(\frac{\partial z}{\partial y}\right)_x$

Define new function

$$w(u, y) \Rightarrow dw = \left(\frac{\partial w}{\partial u}\right)_y du + \left(\frac{\partial w}{\partial y}\right)_u dy$$

For $w = z - ux$, $dw = dz - udx - xdu = udx + vdy - udx - xdu$

$$dw = -xdu + vdy \quad \Rightarrow \left(\frac{\partial w}{\partial u}\right)_y = -x \quad \left(\frac{\partial w}{\partial y}\right)_u = \left(\frac{\partial z}{\partial y}\right)_x = v$$

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For thermodynamic functions:

Internal energy: $U = U(S, V)$
 $dU = TdS - PdV$
 $dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV$
 $\Rightarrow T = \left(\frac{\partial U}{\partial S}\right)_V \quad P = -\left(\frac{\partial U}{\partial V}\right)_S$

Enthalpy: $H = H(S, P) = U + PV$
 $dH = dU + PdV + VdP = TdS + VdP = \left(\frac{\partial H}{\partial S}\right)_P dS + \left(\frac{\partial H}{\partial P}\right)_S dP$
 $\Rightarrow T = \left(\frac{\partial H}{\partial S}\right)_P \quad V = \left(\frac{\partial H}{\partial P}\right)_S$

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Name	Potential	Differential Form
Internal energy	$E(S, V, N)$	$dE = TdS - PdV + \mu dN$
Entropy	$S(E, V, N)$	$dS = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN$
Enthalpy	$H(S, P, N) = E + PV$	$dH = TdS + VdP + \mu dN$
Helmholtz free energy	$F(T, V, N) = E - TS$	$dF = -SdT - PdV + \mu dN$
Gibbs free energy	$G(T, P, N) = F + PV$	$dG = -SdT + VdP + \mu dN$
Landau potential	$\Omega(T, V, \mu) = F - \mu N$	$d\Omega = -SdT - PdV - Nd\mu$

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Lagrangian pictureFor independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

⇒ Second order differential equations for $q_\sigma(t)$ **Switching variables – Legendre transformation**Define: $H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \quad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

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Hamiltonian picture – continued

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \quad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \quad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

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