

## PHY 711 – Contour Integration

These notes summarize some basic properties of complex functions and their integrals. An *analytic* function  $f(z)$  in a certain region of the complex plane  $z$  is one which takes a single (non-infinite) value and is differentiable within that region. Cauchy's theorem states that a closed contour integral of the function within that region has the value

$$\oint_C f(z) = 0. \quad (1)$$

As an example, functions composed of integer powers of  $z$  –

$$f(z) = z^n, \quad \text{for } n = 0, 1, \pm 2, \pm 3, \dots \quad (2)$$

fall in this category. Notice that non-integer powers are generally not analytic and that  $n = -1$  is also special. In fact, we can show that

$$\oint_C \frac{dz}{z} = 2\pi i. \quad (3)$$

This result follows from the fact that we can deform the contour to a unit circle about the origin so that  $z = e^{i\theta}$ . Then

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{e^{i\theta} i d\theta}{e^{i\theta}} = 2\pi i. \quad (4)$$

One result of this analysis is the Cauchy integral formula which states that for any analytic function  $f(z)$  within a region  $C$ ,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z} dz'. \quad (5)$$

Another result of this analysis is the Residue Theorem which states that if the complex function  $g(z)$  has poles at a finite number of points  $z_p$  within a region  $C$  but is otherwise analytic, the contour integral can be evaluated according to

$$\oint_C g(z) dz = 2\pi i \sum_p \text{Res}(g_p), \quad (6)$$

where the residue is given by

$$\text{Res}(g_p) \equiv \lim_{z \rightarrow z_p} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} ((z - z_p)^m g(z)) \right\}, \quad (7)$$

where  $m$  denotes the order of the pole.