Strongly n-trivial Knots

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Abstract

We prove that given any knot k of genus g, k fails to be strongly n-trivial for all n, $n \geq 3g-1$.

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1 Introduction

We start with a little background.

Definition 1.1. *A knot k is called "(n-1)-trivial" if there exists a projection of k, such that one can choose n disjoint sets of crossings of the projection with the property that making the crossing changes corresponding to any of the $2^n - 1$ nontrivial combination of the sets of crossings turns the original knot into the unknot. The collection of sets of crossing changes is said to be an "(n-1)-trivializer for k".*

Conjecture 1.2. *The unknot is the only knot that is n-trivial for all n.*

Note: A knot that is n-trivial is automatically m-trivial for all $m \leq n$.

Work of Gusarov [Gu] and Ng and Stanford [NS] shows that this question equates to showing that the only knot with vanishing Vassiliev invariants for all n is the unknot. Thus, Conjecture 1.2 is at the heart of the study of Vassiliev invariants.

One reason why this question is interesting is that it takes a geometric approach to Vassiliev invariants, instead of the traditional algebraic approach and therefore is relatively unexplored. Vassiliev invariants measure geometric properties of knots, which in turn are geometric objects, so it is reasonable to hope that the geometry might play an integral role in their study.

The following definition derives its motivation from n-triviality.

Definition 1.3. *A knot k is called "strongly $(n$-$1)$-trivial." if there exists a projection of k, such that one can choose n crossings of the projection with the property that making the crossing changes corresponding to any of the $2^n - 1$ nontrivial combination of the selected crossings turns the original knot into the unknot. The collection of crossing changes is said to be a "strong $(n$-$1)$-trivializer for k".*

Note: The expression "n adjacent to the unknot" is used interchangeably with "strongly (n-1)-trivial." We will stick with the latter throughout this paper.

In Section 6 we show that for any n there is a non-trivial knot that is strongly n-trivial. On the other hand in Section 5 we prove the main result of this paper:

Theorem 1.4. *Any non-trivial knot k of genus g fails to be strongly n-trivial for all n, $n \geq 3g - 1$.*

Note: A knot that is strongly n-trivial is automatically strongly m-trivial for all $m \leq n$. Also any knot that is strongly n-trivial is clearly n-trivial, too.

In analogy with Conjecture 1.2 we have

Corollary 1.5. *The unknot is the only knot that is strongly n-trivial for all n.*

Theorem 1.4 is proven by repeated use of the following theorem of Gabai

Theorem 1.6. *(Corollary 2.4 [G]) Let M be a Haken manifold whose boundary is a nonempty union of tori. Let F be a Thurston norm minimizing surface representing an element of $H_2(M, \partial M)$ and let P be a component of ∂M such that $P \cap F = \emptyset$. Then with at most one exception (up to isotopy) F remains norm minimizing in each manifold $M(\alpha)$ obtained by filling M along an essential simple closed curve α in P In particular F remains incompressible in all but at most one manifold obtained by filling P.*

2 Notation

Let k be a knot that is strongly (n-1)-trivial. Let $p : k \to R^2$ be a projection with crossings $\{a_1, \ldots a_n\}$ demonstrating the strong (n-1)-triviality. For each a_i let c_i be the small vertical circle that bounds a disk D_i that intersects k geometrically twice, but algebraically 0 times. We call the c_i linking circles of k and call D_i a crossing disk after [ST]. Let M be the link exterior of $k \cup c_1 \cup \ldots c_n$ and P_i be the torus boundary component in M corresponding to c_i. Either $+1$ or -1 filling of P_i will result in the desired crossing change depending on orientation. We adopt the convention that each P_i will be oriented so that $+1$ filling of P_i corresponds to the appropriate crossing change dictated by a_i.

3 Irreducibility

Lemma 3.1. *Let k be a nontrivial knot. Let $\{c_1, \ldots c_n\}$ be linking circles for k that show k is strongly (n-1)-trivial, then M, the exterior of $k \cup c_1 \cdots \cup c_n$, is irreducible and M is therefore Haken.*

Proof. Assume M is reducible. Let S be a sphere that does not bound a ball on either side. S cannot be disjoint from $D_1 \cup \ldots D_n$ or else it would bound a ball on the side that does not contain k. Assume S intersects $D_1 \cup \cdots \cup D_n$ minimally and transversally. The intersection will consist of a union of circles. Let r be one of these circles that is innermost on S (any circle that bounds a disk on S disjoint from all the other circles of intersection). Without loss of generality assume $r \subset D_1$. r cannot be trivial on $D_1 - (D_1 \cap k)$ since $S \cap D_1$ is minimal. r, however must be trivial in M so must divide D_1 into two pieces, one containing both points of $D_1 \cap k$ and the other consisting of an annulus running from r to c_1. This disk on S bounded by r shows that c_1 bounds a disk in the exterior of k. This, however, means that $+1$ surgery on c_1 leaves k unchanged instead of turning it into an unknot, yielding the desired contradiction. □

4 Minimal genus Seifert surfaces

This section is dedicated to proving the following theorem.

Theorem 4.1. *If k has a strong (n-1)-trivializer $\{c_1, \dots c_n\}$ and F is a Seifert surface for k disjoint from $\{c_1, \dots c_n\}$ which is minimal genus among all such surfaces, then F is a minimal genus Seifert surface for k.*

Proof. Because the c_i have linking number 0 with k we can find a Seifert surface for k disjoint from the c_i. Let F be a minimal genus Seifert surface for k in the link complement.

We supplement the notation introduced in Section 2. Recall M is the link exterior of $k \cup c_1 \cup \cdots \cup c_n$. Let L be the corresponding link of $n+1$ components in S^3. P_i is the torus boundary component in M corresponding to c_i. Let $M(\alpha)$ refer to the manifold obtained by filling M along an essential simple closed curve of slope α in P_n. When $\alpha = 1/m, m \in Z$, $M(\alpha)$ is a link exterior. Let L_α be the corresponding link in S^3. Let k_α be the image of k in L_α and F_α be the image of F in L_α.

We now prove Theorem 4.1 by induction on n. If F is ever a disk then Theorem 4.1 is clearly true, so we will assume that F is not a disk throughout the proof.

The base case: Let k be a strongly 0-trivial knot. This means that k is unknotting number 1 and there is one linking circle c_1 that dictates a crossing change that unknots k.

By Lemma 3.1 if M is reducible, then k is the unknot. As in the proof of Lemma 3.1 c_1 bounds a disk in the complement of k, so $k \cup c_1$ is the unlink on two components. Therefore, F being least genus must be a disk, which is a contradiction, verifying the claim for M reducible and $n = 1$. We may assume M is irreducible to complete the base case. k_1 is the unknot. Since F_1 is not a disk, it is no longer norm minimizing after the filling. Thus by Theorem 1.6 F is norm minimizing under any other filling of P_1. In particular F_∞ is Thurston norm minimizing for L_∞, which is just k. Thus, F is a least genus Seifert surface for k.

The inductive step: Now we assume that if k has a strong (n-2)-trivializer $\{c_1, \dots c_{(n-1)}\}$ and F is a Seifert surface for k disjoint from $\{c_1, \dots c_{n-1}\}$, which is minimal genus among all such surfaces, then F is also a minimal genus Seifert surface for k and show that the same must be true for any strong (n-1)-trivializer for k.

Again by Lemma 3.1 if M is reducible, k must be the unknot. As in previous arguments, the separating sphere S must intersect at least one D_i in a curve that is essential on $D_i - k$. Without loss of generality, we may assume that D_n is such a disk. Then c_n bounds a disk in the complement of $k \cup \{c_1, \ldots c_{n-2}\}$. Since $\{c_1, \ldots c_{n-1}\}$ forms a strong (n-2)-trivializer for k, the induction assumption implies k bounds a disk Δ disjoint from $c_1 \cup \ldots c_{n-1}$. Since c_n bounds a disk disjoint from $k \cup c_1 \cup \ldots c_{n-1}$, Δ can clearly be chosen to be disjoint from c_n, too, but this contradicts the assumption that F was minimal genus, but not a disk.

We now may finish the proof of Theorem 4.1 knowing that M is irreducible. k_1 is an unknot in the link L_1. $\{c_1, \ldots c_{n-1}\}$ is a strong (n-2)-trivializer for k in L_1. The inductive assumption means that k_1 bounds a disk in the exterior of L_1. This disk is in the same class as F_1 in $H_2(M(1), \partial M(1))$, thereby showing that F_1 is not Thurston norm minimizing. Thus, by Theorem 1.6 F remains norm minimizing under any other filling of P_n. In particular F_∞ is Thurston norm minimizing in L_∞. Thus, F is a least genus Seifert surface for k in the complement of $\{c_1, \ldots c_{n-1}\}$. $\{c_1, \ldots c_{n-1}\}$, however, forms a strong (n-2)-trivializer for k in L_∞. By the inductive assumption, F must be Thurston norm minimizing for k in the knot complement as well as the link complement. □

5 Arcs on a Seifert surface

We now prove Theorem 1.4: Any non-trivial knot k of genus g fails to be strongly n-trivial for all n, $n \geq 3g - 1$.

Proof. Let k be strongly n-trivial with n-trivializers $\{c_1, \ldots c_{n+1}\}$. Let F be a minimal genus Seifert surface for k disjoint from $\{c_1, \ldots c_{n+1}\}$ as in Theorem 4.1. F has genus g.

Each linking circle c_i bounds a disk D_i that intersects F in an arc running between the two points of $k \cap D_i$ and perhaps some simple closed curves. Simple closed curves inessential in $D_i - k$ can be eliminated since F is incompressible. Any essential curves s_j must be parallel to c_i in $D_i - k$. These curves can be removed one at a time using the annulus running from c_i to the outermost s_j to reroute F, decreasing the number of intersections. Thus, if F is assumed to have minimal intersection with each of the D_i then it intersects each one in an arc which we shall call a_i as in Figure 1. Each a_i is essential in F. Otherwise c_i would bound a disk disjoint from F and the crossing change along c_i would fail to unknot k.

Lemma 5.1. *a_i is never parallel on F to a_j for $i \neq j$.*

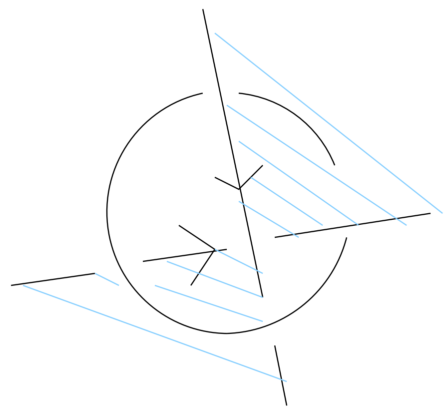

Figure 1: A Seifert surface passing disjointly through a linking circle

Proof. If a_i is parallel on F to a_j there must be an annulus running from P_i to P_j in the link exterior. Recall that we adopted the convention that P_i and P_j are each oriented so that $+1$ surgery results in the appropriate crossing changes. The two tori cannot have opposite orientations or else $+1$ fillings on both P_i and P_j is the same as ∞ fillings on both, thus, instead of unknotting k changing both crossings leaves k knotted. If the two tori have the same orientation we could replace $P_i \cup P_j$ with a single torus T obtained by cutting and pasting of the two tori along the annulus. Now $+1$ filling for P_i and ∞ filling for P_j is equivalent to $+1$ filling on T, while $+1$ filling on both P_i and P_j is equivalent to $\frac{1}{2}$ filling on T. This implies that F fails to be norm-minimizing after both $+1$ and $\frac{1}{2}$ filling of T. This contradicts Theorem 1.6 completing the proof of the Lemma. □

Then $\{a_1, \ldots a_n\}$ is a collection of embedded arcs on F, no two of which are parallel. An Euler characteristic argument shows that $m \leq 3g-1$. Since the arcs are in one to one correspondence with the linking circles, a strong n-trivializer can produce at most $3g-1$ linking circles for k finally proving Theorem 1.4. □

We note that Theorem 1.4 predicts that a genus one knot can be at most strongly 1-trivial. Given standard projections of the trefoil and the figure eight knot it is easy to find a pair of crossing changes that show the knots are strongly 1-trivial. The theorem is therefore sharp at least for genus one knots. It is possible, but unlikely, that the theorem remains as precise for higher genus knots.

Finally as noted in the introduction, Theorem 1.4 implies Corollary 1.5: The unknot is the only knot that is strongly n-trivial for all n.

6 Constructing strongly n-trivial knots

One might think that there exists a bound n such that no nontrivial knot is strongly n-trivial. Given any n, this section gives one way to construct strongly n-trivial knots.

In Figure 3 we will give projections of graphs that show how to turn an unknot into a strongly n-trivial knot. The circle running around the outside of the graph should be viewed as an unknot k'. Each arc a_i suggests a linking circle c_i and a crossing disk D_i. If we alter the link $k \cup c_1 \cup \cdots \cup c_n$ in S^3 by twisting -1 times along each of the disks $D_1, D_2, \ldots D_n$ k' becomes a new knot k see Figure 2. The linking circles remain fixed, so we get a new link in S^3, $k \cup c_1 \cdots \cup c_n$.

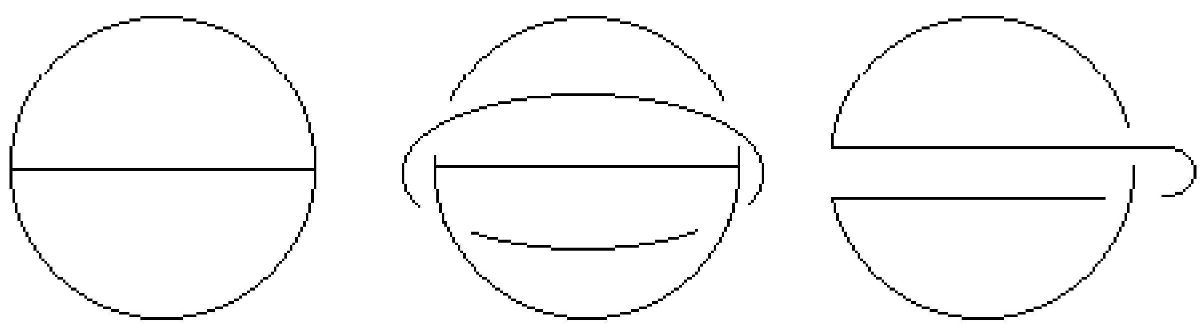

Figure 2: A graph contains instructions for turning the unknot into a knot (or perhaps another unknot).

Figure 3 gives graphs that generate examples of strongly 1-trivial and strongly 2-trivial knots. Note that the figure on the right is obtained from the figure on the left by replacing one arc by two new arcs that follow along the original arc, clasp, return along the original arc, and then, close to the boundary, clasp once again. This process could be iterated indefinitely by choosing an arc of the new graph and repeating the construction. It is modeled on doubling one component of a link. Given a Brunnian link of n components (a nontrivial link for which any $n-1$ components is the unlink), doubling one of the components yields a Brunnian link of $n+1$ components. The graph on the left in Figure 3 has a Brunnian link of 2 components as a subgraph and the one on the right has the double of that link as a subgraph. Let Γ_n be the graph after $n-2$ iterations ($n \geq 2$).

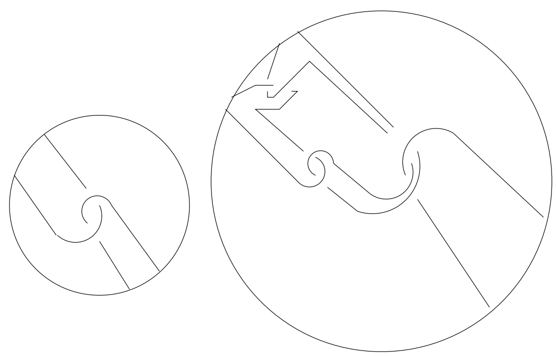

Figure 3: Examples of crossing changes for the unknot that create nontrivial knots that are strongly 1-trivial (left) and strongly 2-trivial (right). Note that each contains a subgraph that is a Brunnian link of $n+1$ components.

Theorem 6.1. Γ_n *contains a Brunnian link,* l_{n+2}*, of* $n+2$ *components and yields* k *a non-trivial, strongly* $(n+1)$*-trivial knot.*

Proof. The link consists of the arcs $\{a_1, \ldots a_{n+2}\}$, together with short segments from k' connecting the end points of the segments (and disjoint from the end points of the other segments). The base case is trivial because, Γ_0 contains a Brunnian link of 2 components: the Hopf link. Γ_n is obtained from Γ_{n-1} by doubling one of the components of a Brunnian link of $n+1$ components. This yields a Brunnian link of $n+2$ components.

As a result of the Brunnian structure in Γ_n any $n+1$ edges from $\{a_1, \ldots a_{n+2}\}$ can be disjointly embedded on a disk bounded by k'. So k' forms an unlink with any proper subset of $\{c_1, \ldots c_{n+2}\}$.

We can use this fact to show that $\{c_1, \ldots c_{n+2}\}$ are an n-trivializer for k. Let S be any nontrivial subset of $\{c_1, \ldots c_{n+2}\}$. Let S^c be the complement of S. If we take k together with S, and do $+1$ surgery on each component of S

the resulting knot is an unknot. This is because it is exactly the same as if we took k' and did -1 surgery on each of the components of S^c. Since S is a nontrivial subset, S^c is a proper subset. k' together with a the linking circles in S^c, therefore form an unlink, so each of the components of S^c bounds a disk disjoint from k' and doing -1 surgery on these linking circles leave k' unchanged.

Now that we know that k is strongly $(n+1)$-trivial, we need only show k is a non-trivial knot. Assume k is trivial. By Theorem 4.1, k bounds a disk Δ in the complement of $c_1 \cup \cdots \cup c_{n+2}$. Since $k \cup c_1 \cup \ldots c_{n+2}$ was obtained from $k' \cup c_1 \cup \ldots c_{n+2}$ by spinning along the D_i's, the exteriors of the two links are homeomorphic, and therefore k' must bound a disk Δ' also disjoint from $c_1 \cup \cdots \cup c_{n+2}$ (note that one could even prove that both $k \cup c_1 \cup \cdots \cup c_{n+2}$ and $k' \cup c_1 \cup \cdots \cup c_{n+2}$ are unlinks). k' intersects each D_i in 2 points, so as before we may assume $\Delta' \cap D_i$ is an arc for each i, but these arcs must, of course, be isotopic to the $a_i s$ which in turn shows that the $a_i s$ can be disjointly embedded on Δ', proving that l_{n+2} is planar and not Brunnian, the desired contradiction. Thus, k is a strongly $(n+1)$-trivial knot, but not the unknot.

□

7 References

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