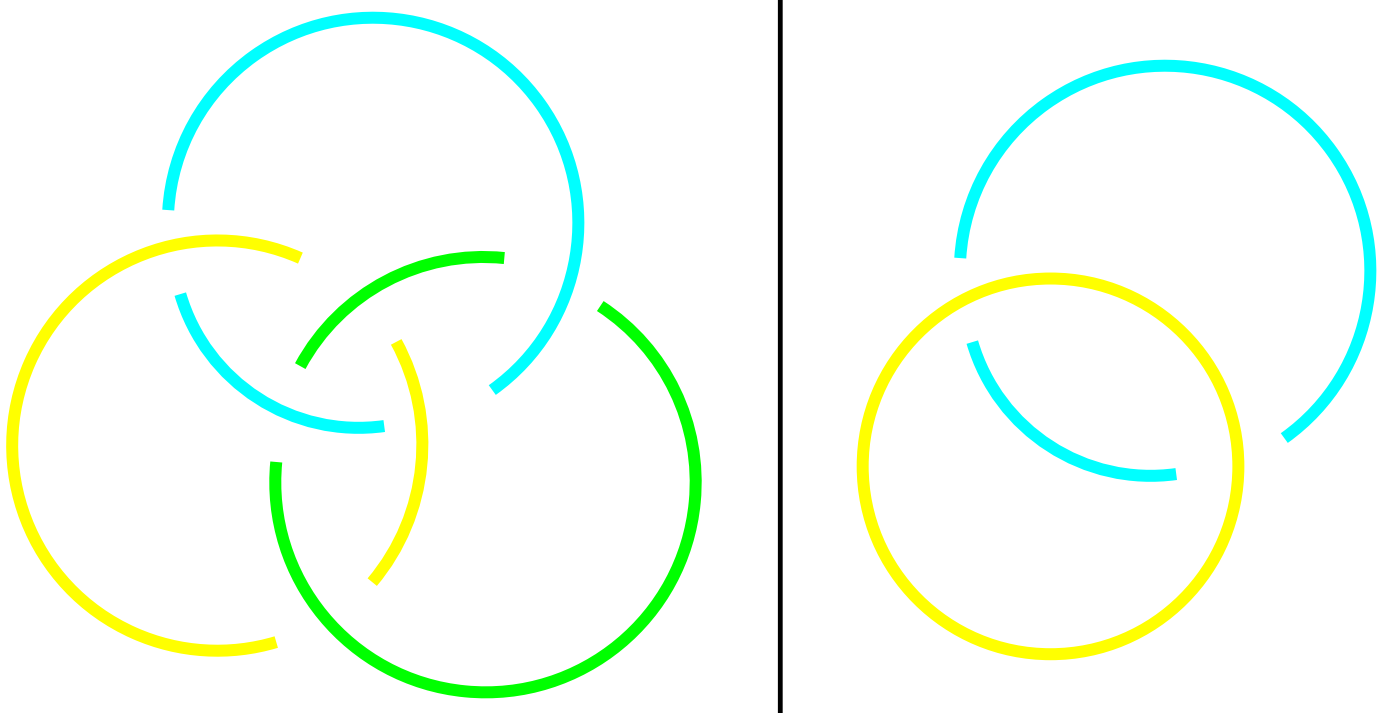


The Trouble with Brunnian Circles

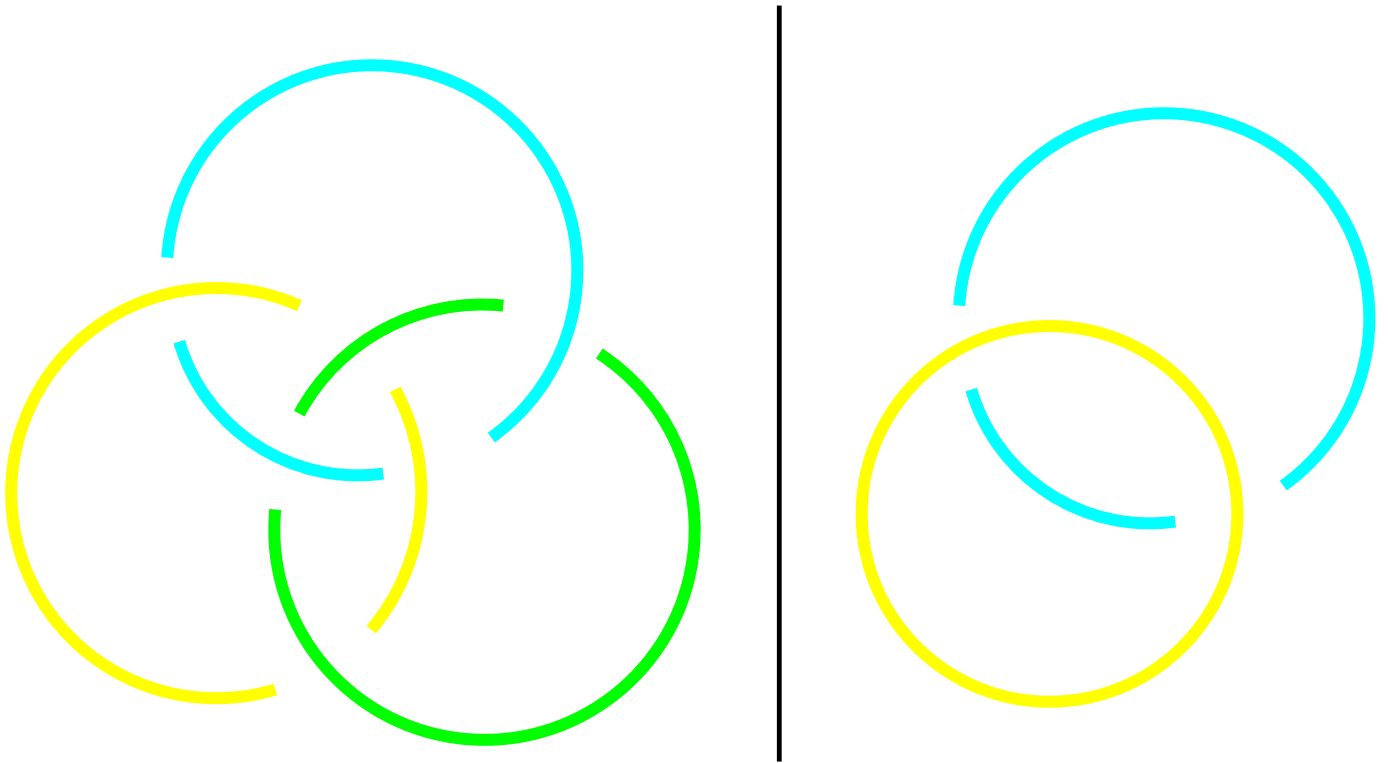


Hugh Nelson Howards

- What is a knot?
- What is the unknot?
- What is a link?
- What is a split link?
- What is an unlink?

Brunnian Circles

Definition 1. *A Brunnian link is a link with $n \geq 3$ components, such that the entire link is not the unlink, but every link of $n - 1$ components or fewer is the unlink.*



The Borromean Rings are a Brunnian link of 3 components

props to erin and bob come on over the
rest of you!!!!

Theorem 2. (*Freedman, Skora*) *No Brunnian Link can be built out of (geometrically) round circles.*

- This was proven in the 80's by Mike Freedman.
- It was reproven just for the case of the Borromean Rings in the early 90's by Bernt Lindström and Hans-Olov Zetterström (in Borromean circles are impossible. *Amer. Math. Monthly* 98 (1991), no. 4, 340-341.) and
- independently the case of the Borromean rings was reproven in the simplest proof yet by Ian Agol in 1993.

Theorem 3. *The only Brunnian link of $n \leq 4$ components that can be built out of convex planar curves are the Borromean Rings.*

- The Borromean rings can be built from 2 circles and an ellipse.
 - Let one circle be $x^2 + y^2 = 9, z = 0,$
 - the second be $y^2 + z^2 = 16, x = 0$
 - and the ellipse be the curve $\frac{x^2}{4} + \frac{z^2}{25} = 1, y = 0.$
- There are an infinite number of Brunnian links with 3 components. (the same is true for any $n, n \geq 3$).
- Every Brunnian link of n components can be built out of $n - 1$ components that are circles and 1 component that is not.

Some motivation for studying the Borromean
Rings ...

A Second generalization: We also generalize Freedman's theorem to higher dimensions.

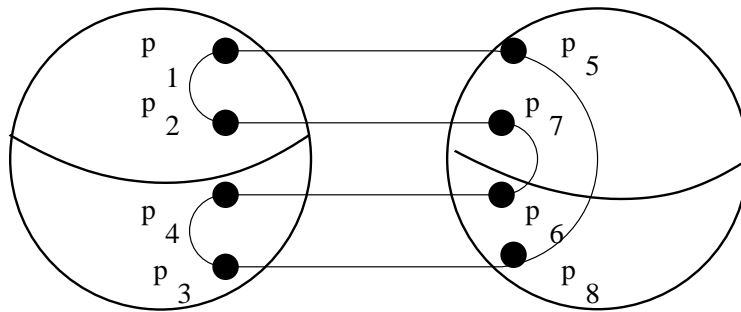
A subset K of R^n is a generalized knot, if K is homeomorphic to a sphere S^p . L is a generalized link if L is homeomorphic to a disjoint union $S^{p_1}, S^{p_2}, \dots, S^{p_r}$ of one or more spheres (possibly of different dimensions).

L is said to be a generalized unlink if for each of its components bounds a ball disjoint from all the other components. If we set, $n = 3$ and look require all the spheres to be 1-spheres (ie circles) we, of course, have the traditional definition of an unlink.

If a (generalized) link L of three or more components is not a (generalized) unlink, yet every proper sublink is a (generalized) unlink, we say L is a (generalized) Brunnian link.

Theorem 4. *No generalized Brunnian Link can be built out of (geometrically) round spheres.*

Here is an example of a generalized Brunnian link.



The projection of a Brunnian link in R^4 into R^3 by Π_ω

Let $x, y, z, w \in R^4$. Let Π_ω project R^4 onto R^3 via the map $\Pi_\omega(x, y, z, w) = (x, y, z)$.

This is a Brunnian link if before projecting into R^3 by Π_ω the w component of both spheres is always 0 and the w component of the knot is -1 for p_i with i odd, and $+1$ for p_i with i even.

Question 5. *What happens if you are wearing Brunnian clothes and you take off any piece of your outfit?*

Theorem 6. *You run quickly from the room blushing.*

Proof. The Proof is left as an exercise for the audience (to try at home). □

A couple open questions to ponder (perhaps an undergraduate or masters student could extend this argument to solve one of these questions or solve it with his or her own techniques)

Open Question 7. *Can a Brunnian link be formed from 6 or more convex planar components?*

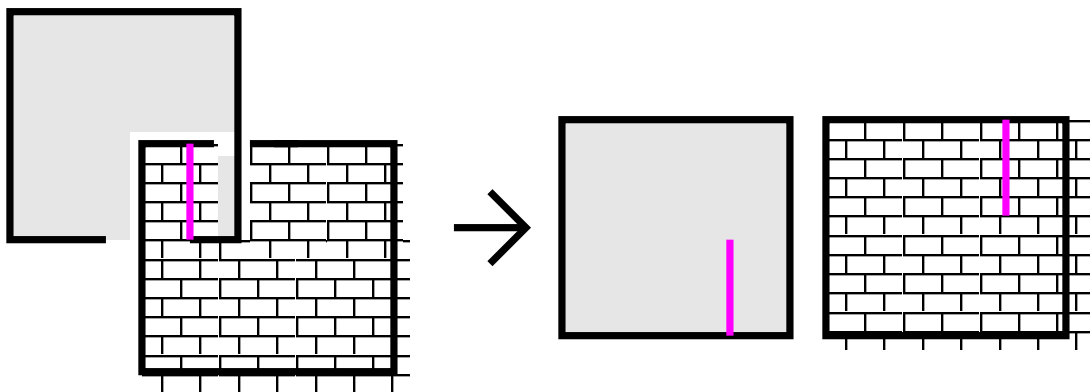
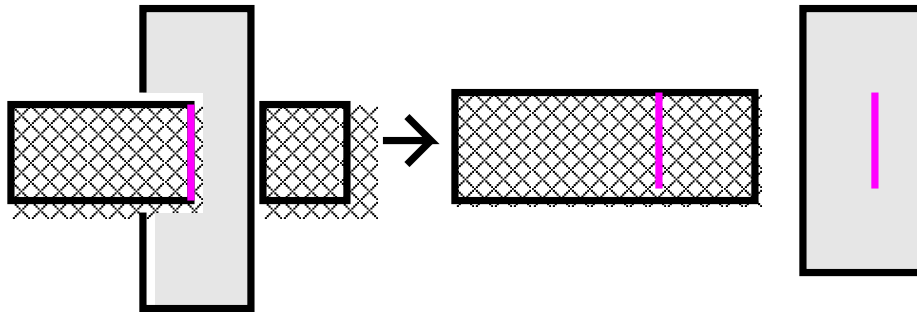
Open Question 8. *Can a generalized Brunnian link (other than the Borromean Rings) ever be formed from convex components?*

Note: I am happy to provide a copy of this preprint for anyone who wants to work on either of these questions!

The case of the convex three component link.

Let $L = s_1 \cup s_2 \cup s_3$ be a link consisting of three convex, planar curves, bounding planar disks Δ_1, Δ_2 , and Δ_3 respectively. Assume we do not have the Borromean rings. We would like to prove that L is the unlink.

How can two convex, planar disks intersect?

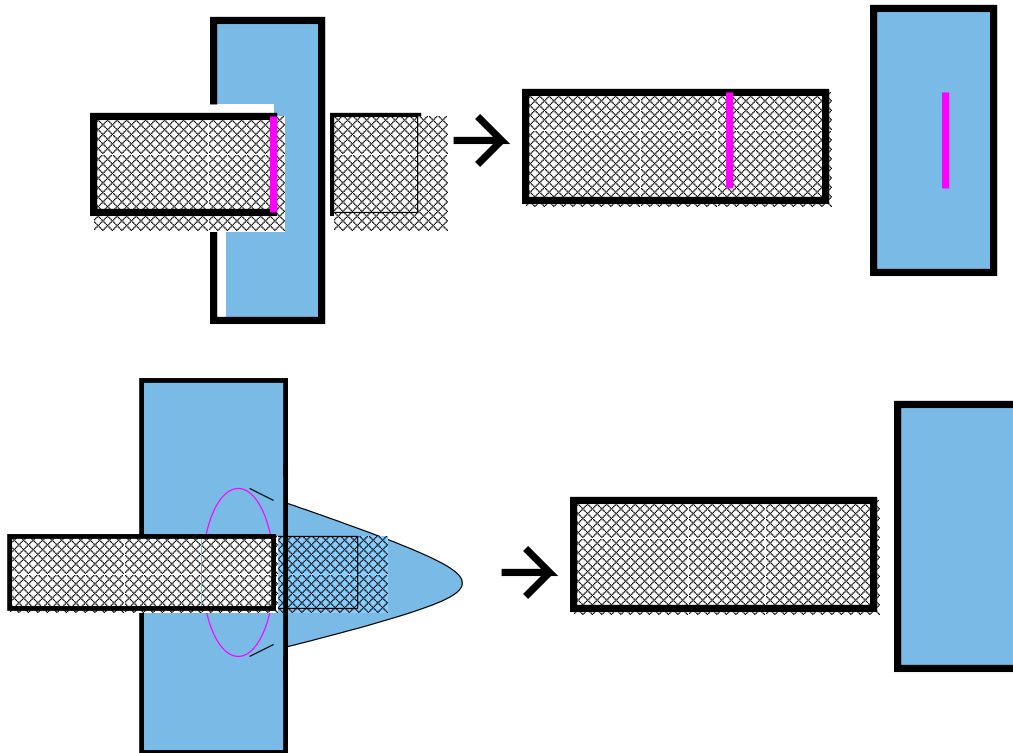


There are two possible types of intersections but the second type can only happen when two of the components are linked together and thus only the first kind of intersection can happen in a Brunnian link.

Lemma 9. *No component of a Brunnian link can bound a disk disjoint from the other components of the link, and thus no disk can have only exterior arcs for its intersection pattern.*

Proof. If one disk has only exterior arcs on it, then its boundary is a knot that is not linked with the other components (and thus the link can't be Brunnian). \square

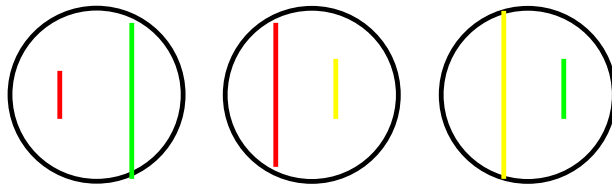
Case 0 : The three disks planar disks that the components of L bound intersect in exactly 0 triple points. In this case, the worst scenario is that all the disks intersect each other and thus each disk has two arcs of intersection on it since there are three ways to pair up the disks (and thus at most three intersections) so there are at most three interior arcs, and at most three exterior arcs.



Sometimes you can eliminate intersections by “pushing” one of the disks off of the other (the new disk is not flat, but the resulting disks are disjoint)

We argue that if there are no triple points we are in the situation above (the same argument would not work for a six component link without triple points). Once the disks are disjoint, Lemma 9 shows that the link was not Brunnian, finishing off the case of 0 triple points.

By Lemma 9 no component has two exterior arcs on it so each component must have one interior arc and one exterior arc on it.

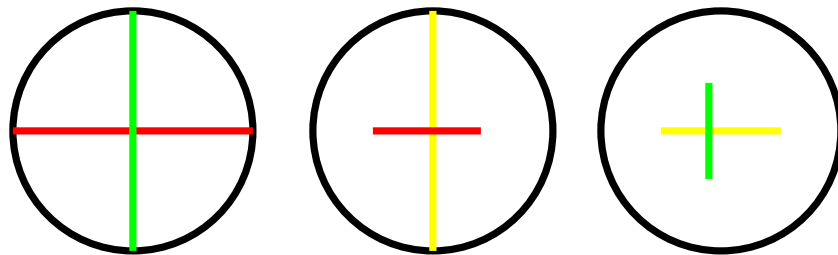


A Possible intersection pattern if there are no triple points

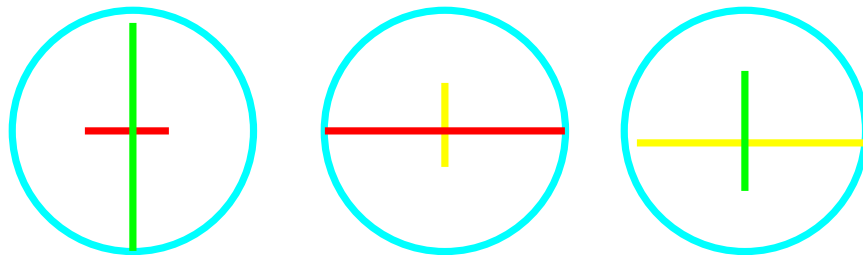
If we see this pattern we can manipulate the disks to get rid of all of the intersections and thus we must have the unlink!

Case 1 : The disks intersect in exactly one triple point. Even in this case, by Lemma 9 no component has two exterior arcs on it so each component must have one interior arc and one exterior arc on it as in Figure B below.

A



B



The only possible intersection patterns if there is one triple point. (A must be the unlink and B must be the Borromean rings)

Lemma 10. *The Borromean Rings, are the unique link with three convex planar components, bounding planar disks, with one triple point in their intersection and one exterior arc and one interior arc on each of the three disks.*

Thus, if L is not the unlink it must be the Borromean Rings!

A sample of the proof for four components

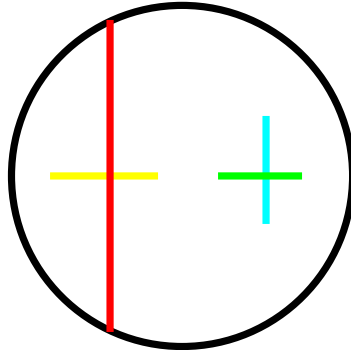
Let $L = s_1 \cup s_2 \cup s_3 \cup s_4$ be a link consisting of four convex, planar curves. Since we have four components we do not have the Borromean rings. We would like to prove that L is the unlink.

Case 0: The disks intersect in 0 triple points. This argument is much like the argument in the case of 3 components.

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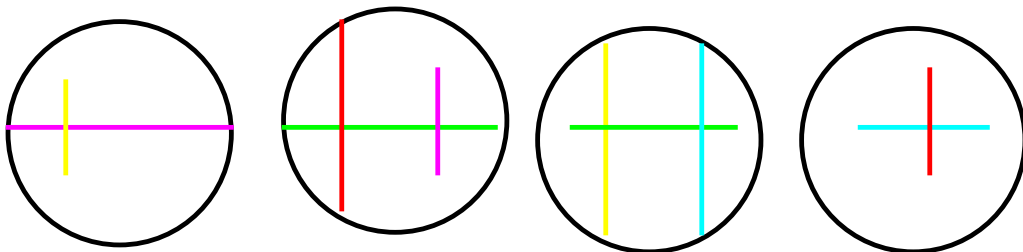
Case 2: The disks intersect in exactly 2 triple points.

Since we have four disks, each disk can have at most 3 intersection arcs on it (one for each of the other three disks). This means we cannot see something like the figure below



We don't have enough intersection arcs to see this.

So we must see something more like the following figure.



This figure cannot be a Brunnian link because the yellow purple and green intersection arcs show that it contains the Borromean Rings as a sublink (and for a Brunnian link every sublink is an unlink).

A careful analysis shows that every possible way to draw the intersection arcs either gives an unlink or has this problem!

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