

# Brunnian spheres

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## 1 Introduction

Is the link in Figure 1 built out of round circles?

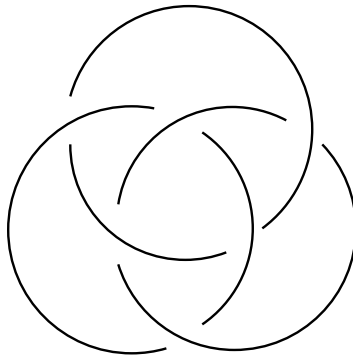


Figure 1: The Borromean Rings are a Brunnian link.

Surprisingly this link appears to be built out of 3 round circles, but a Theorem of Mike Freedman and Richard Skora (Theorem 2.1) proves that this must be an optical illusion!

In this paper we bring to light and extend this theorem that showed no Brunnian link can be built out of round components. We address two different traditional generalizations of Brunnian Links.

Recall that a knot is a subset of  $R^3$  (or of the 3-sphere  $S^3$ ) that is homeomorphic to a circle (also called a 1-sphere or  $S^1$ ). Informally a knot is said to be an unknot if it can be deformed through space to become a perfect (round) circle without ever intersecting or crossing itself (see Figure 2) otherwise it is said to be knotted (see Figure 3).

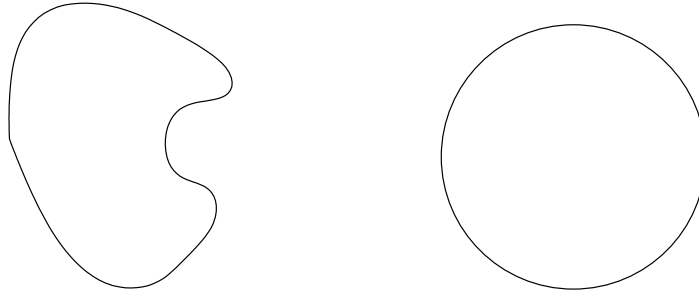


Figure 2: The figure on the left is an unknot because it can be straightened to look like the figure on the right without introducing any self intersections.

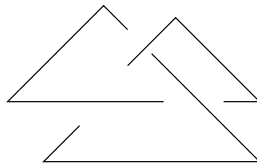


Figure 3: The trefoil, pictured above, is not an unknot because to deform it to a round circle it must first cross itself.

A link,  $L$ , is just a collection of disjoint knots.  $L$  is an unlink of  $n$  components, if it consists of  $n$  unknots and all of the components can be separated from each other without passing through each other (more rigorous definitions are given in Section 3). Figure 2 could be thought of as an unlink of 2 components. Figure 4 is the Hopf link, the simplest 2 component link that is not an unlink. Figure 5 is an unlink of 3 components.

A link,  $L$ , is said to be Brunnian if  $L$  is a link of  $n \geq 3$  components such that  $L$  is not the unlink of  $n$  components, but every proper sublink of  $L$  is an unlink. The most famous Brunnian link is called the Borromean rings, pictured in Figure 1.

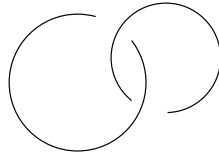


Figure 4: The Hopf link

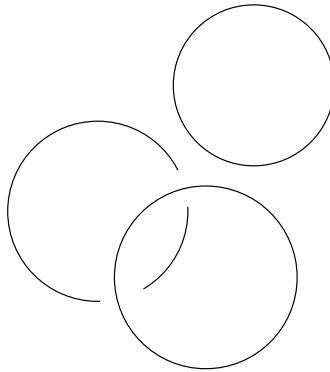


Figure 5: An unlink

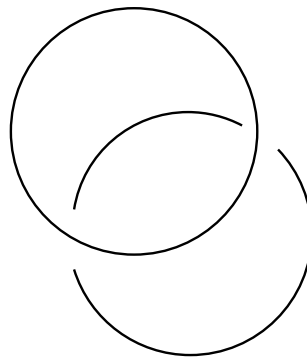


Figure 6: The Borromean Rings become an unlink if any component is deleted (here we have deleted the bottom left component, but any component would have yielded the same result)

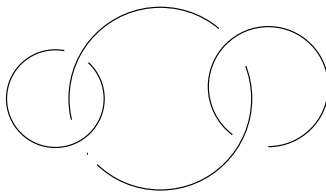


Figure 7: This is not an unlink, but it is not Brunnian either (if you remove the component in the center it becomes an unlink, but if you remove the component on the right instead, it does not.)

## 2 A little historical context

Brunnian links were introduced over 100 years ago when knot theory was a relatively new field of study. Brunn introduced them in his 1892 paper “Uber Verkettung” (On Linking) [B].

In 1961 Debrunner [De] more rigorously proved some of the claims from [B] using the tools and language of modern knot theory. Debrunner also generalized Brunnian links by defining  $(m, k)$ -Brunnian links.

A link is an  $(m, k)$ -Brunnian link if it has  $m$  components, for which no sublink of  $k$  or more components is an unlink, but every sublink of  $\leq (k - 1)$  components is an unlink. Further work on these links were done by D. Penney in 1961 [P].

T. Yanagawata also looked at a different generalization of Brunnian links in 1969. He looked at the question in higher dimensions and gave examples of 2-spheres in  $R^4$  that formed Brunnian Links [T].

In 1987 Freedman and Skora proved the following theorem in their paper “Strange actions of groups on spheres” in the Journal of Differential Geometry [FS].

**Theorem 2.1.** *(Freedman, Skora) No Brunnian Link can be built out of round circles.*

The special case of the Borromean Rings was reproven in the early 90’s by Bernt Lindstrom and Hans-Olov Zetterstrom in [LZ], and independently (but unpublished) in the simplest proof yet by Ian Agol in 1993. Lindstrom and Zetterstrom did not know about Freedman and Skora’s proof when they published their more complicated and less general proof four years later.

Our primary generalization of Freedman and Skora follows.

**Theorem 2.2.** *No Brunnian Link in  $R^n$  (or  $S^n$ ) can be built out of round spheres.*

It is also easy to generalize it in the following manner.

**Corollary 2.3.** *No  $(m, k)$ -Brunnian link in  $R^n$  (or  $S^n$ ) can be built out of round circles (spheres) if  $k > 2$ .*

Note that the definition of an  $(m, k)$  Brunnian link does not make sense for  $k < 2$  and is not interesting if  $k = 2$  since the second condition reduces to just requiring that the components of the link are unknots (and thus trivially 1 component unlinks). Figure 8 show a  $(3, 2)$ -Brunnian link built out of round circles. This time it is not hard to confirm that it is not an optical illusion. The link is a generalization of a Brunnian link, but is not itself a Brunnian link, so we do not have a contradiction to Freedman and Skora's theorem.

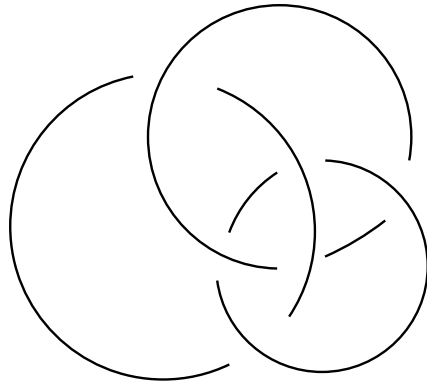


Figure 8: A  $(3, 2)$ -Brunnian link made out of three round circles.

A third, partial generalization appears in [H], which merely requires that the link components be convex curves (curves that bound convex, planar regions) instead of round circles, and concludes:

**Theorem 2.4.** [H] *The Borromean rings are the unique Brunnian links of three or four components that can be formed out of convex curves.*

Robert Davis has since extended this result further to prove that

**Theorem 2.5.** [D] *No Brunnian link of five components can be formed out of convex curves.*

We make a few observations about Brunnian links. For each  $n, n \geq 3$  there are an infinite number of Brunnian links with  $n$  components. One four component Brunnian link is pictured in Figure 9. We see that if any one component is deleted, the others can be pulled apart one component at a time. Note also the 4-fold symmetry. It is easy to use this symmetrical structure to build a Brunnian link of  $n$  components.

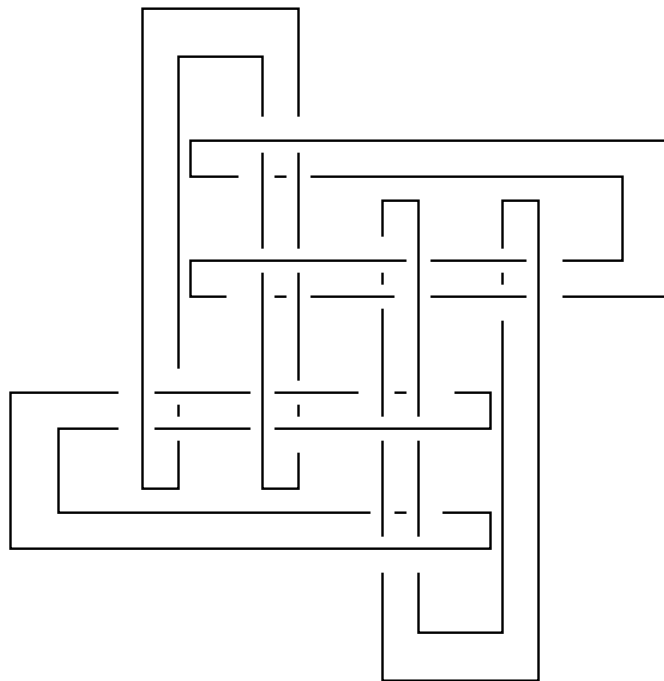


Figure 9: A Brunnian link with four components

Another, more flexible, method of construction is to take the infinite family of non-trivial links made up of two unknotted components and take the Bing double of one of the components. (See Figure 10). This yields an infinite family of three component Brunnian links. If we double one of the components of each of the new links we get an infinite family of four component Brunnian links. We can, of course, continue this process as long as we like.

The author would like to thank Ian Agol and Seungsang Oh for helpful comments.

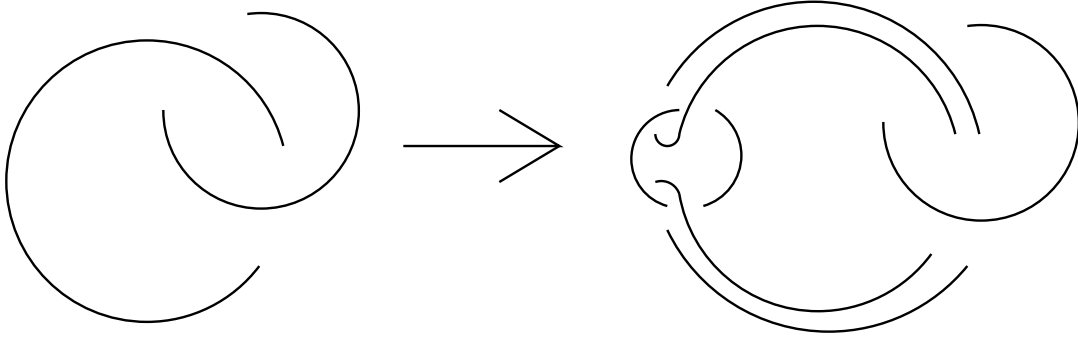


Figure 10: A Bing double of the Hopf Link yields a Brunnian Link. (It is, in fact, a less conventional picture of the Borromean rings)

### 3 Definitions

We turn to some standard definitions.

- Let  $\underline{R}^n$  be Euclidean  $n$ -space with the traditional notion of distance on it.
- The unit n-ball  $\underline{B}^n$  in  $R^n$  is all points of distance less than or equal to one from the origin (thus the unit 1-ball is an interval, the unit 2-ball is a disk, and so on).
- The unit n-sphere (or  $\hat{S}^n$ ) in  $R^{n+1}$  is all points exactly distance one from the origin (thus, the unit 1-sphere is a circle etc.).
- A subset  $F$  of  $R^n$  is called a k-sphere (or  $S^k$ ), if  $F$  is homeomorphic to the unit k-sphere.
- A subset  $B$  of  $R^n$  is a k-ball (or  $B^k$ ), if  $B$  is homeomorphic to the unit k-ball.
- We say a subset  $F$  of  $R^n$  is a round sphere if  $F$  can be described by a formula of the form  $(x_i - c_i)^2 + (x_j - c_j)^2 + \dots + (x_k - c_k)^2 = r^2$  ( $1 \leq i < j \dots < k \leq n$ ), and  $x_t = c_t$  for each  $t, 1 \leq t \leq n$ , and  $t \notin \{i, j, \dots, k\}$ . Where  $c_s$  is a constant, for every  $s, 1 \leq s \leq n$ .

Thus, the ellipse  $\frac{x_1^2}{4} + \frac{x_2^2}{9} = 1 \subset R^2$  is a 1-sphere or (circle), but it is not a “round 1-sphere,” or (round circle).

All properties of knots and links that we are interested in here are preserved by conformal mappings between  $R^n$  and  $\hat{S}^n$  since conformal mappings take circles to circles. Linking properties that hold in  $R^n$  will also hold in  $S^n$ . Because the proofs are slightly easier in  $S^n$ , we will think of our links as subsets of  $\hat{S}^n$  for the rest of the paper.

- We say  $K$ , a subset of  $\hat{S}^n$ , is a knot in  $\hat{S}^n$ , if  $K$  is homeomorphic to a  $p$ -sphere, ( $p < n$ ).
- $L$  is a link in  $\hat{S}^n$  if  $L$  is homeomorphic to a disjoint union of one or more knots (possibly of different dimensions).

Note that knot theory is usually restricted to the case where  $\hat{S}^n$  is  $\hat{S}^3$  (or  $R^3$ ) and each of the knots are homeomorphic to 1-spheres. This situation will be designated by the terms knots and links (as opposed to knots in  $\hat{S}^n$  and links in  $\hat{S}^n$ ).

- Let  $L = F_1 \cup F_2 \cup \dots \cup F_m \subset \hat{S}^n$ .  $L$  is said to be an unlink in  $\hat{S}^n$  if for each  $i$ ,  $F_i$ , bounds a ball  $B_i \subset \hat{S}^n$  such that  $B_i \cap F_j = \emptyset$  for all  $j \neq i$ .

Again, if we set,  $n = 3$  and restrict to circles, we, of course, have the traditional definition of an unlink, where each component bounds a disk disjoint from the other components.

- If a link  $L$  (in  $\hat{S}^n$ ) of  $m \geq 3$  components is not a unlink (in  $\hat{S}^n$ ), yet every proper sublink is an unlink (in  $\hat{S}^n$ ), we say  $L$  is a Brunnian link (in  $\hat{S}^n$ ).
- If a link  $L$  in  $\hat{S}^n$  of  $m \geq 3$  components has no sublink of  $k$  or more components that is an unlink in  $\hat{S}^n$ , but every sublink of fewer than  $k$  components is an unlink in  $\hat{S}^n$ , we say  $L$  is an (m,k) Brunnian link (in  $\hat{S}^n$ ).

## 4 Intersection of Flat Balls

With terminology in place, we now begin the proof of Theorem 2.2

Assume  $L = F_1 \cup F_2 \cup \dots \cup F_k$  is a Brunnian Link made of round spheres in  $\hat{S}^{(n-1)}$ . As usual we can assume  $\hat{S}^{(n-1)} \subset \hat{B}^n \subset R^n$ . We will examine what happens if  $L$  contained two 2-spheres, say  $F_1$  and  $F_2$ . The other cases are analogous.

Let  $X_1$  be a (flat) copy of  $R^3 \subset R^n$  such that  $X_1 \cap \hat{S}^{(n-1)} = F_1$ , and let  $X_2$  be a (flat) copy of  $R^3 \subset R^n$  such that  $X_2 \cap \hat{S}^{(n-1)} = F_2$ . Let  $D_1 = X_1 \cap \hat{B}^n$  and  $D_2 = X_2 \cap \hat{B}^n$ .  $D_1$  is a 3-ball bounded by  $F_1$  (in  $R^n$ , not in  $\hat{S}^{(n-1)}$ ), and  $D_2$  is a 3-ball bounded by  $F_2$  (again in  $R^n$ , not in  $\hat{S}^{(n-1)}$ ).

Note: while  $R^n$  is hard to picture for  $n \geq 4$ , we can get an intuitive picture of  $X_i, X_j, F_i, F_j, D_i, D_j$ , and  $(D_i \cap D_j)$  by looking down a few dimensions. In Figure 11, we let  $F_i$  be a 1-sphere,  $F_j$  be a 0-Sphere (2 points) both contained in the unit 2-Sphere in  $R^3$ . In that context,  $X_i$  is a plane,  $X_j$  is a line,  $D_i$  is a disk,  $D_j$  is an interval, and  $(D_i \cap D_j)$  is a point.

Returning to the case of two 2-spheres in  $\hat{S}^{(n-1)}$   $X_1 \cap X_2 = X_3$  will be either

1. The null-set,
2. a point,
3. a line, (a copy of  $R^1$ ) or
4. a plane (a copy of  $R^2$ )

This implies that  $D_1 \cap D_2 = D_3$  is either

1. The null-set,
2. a point,
3. an interval, or
4. a disk.

This final two options, are, of course, impossible since  $D_3 \subset D_1 \cap D_2$  and  $\partial D_3 \subset \partial D_1 \cap \partial D_2 = F_1 \cap F_2 = \emptyset$ . Thus the intersection of  $D_1$  and  $D_2$  can have no boundary and must be either the null-set or a point.

By general position if  $D_3$  above is a point, then  $n = 6$  since two 3-balls intersecting in general position in  $R^n$  will always be disjoint in  $n > 6$ , and will always be disjoint or intersect in more than a point if  $n < 6$ . We would like to show that  $D_3$  must be the null-set, so let us assume that  $n = 6$ , the only troublesome case.

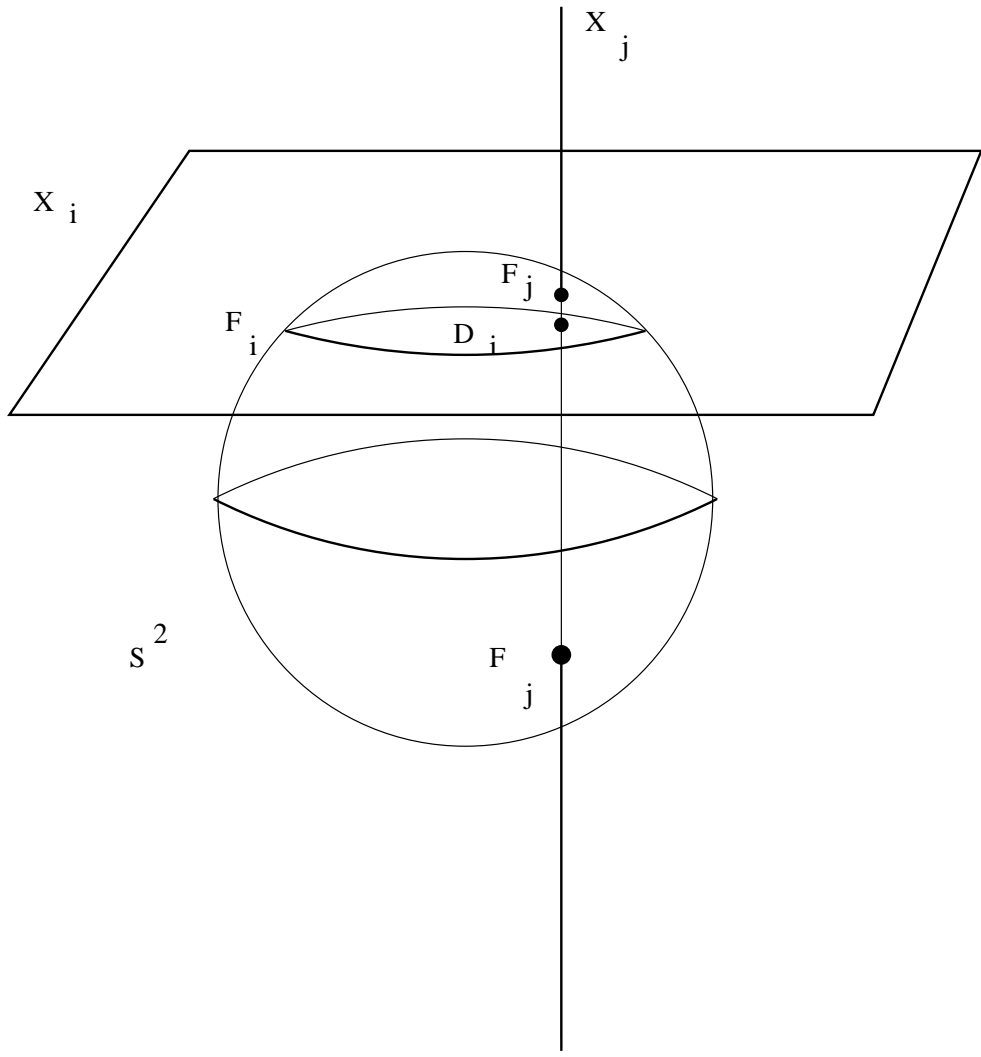


Figure 11:  $F_i$  is a 1-sphere linked in  $\hat{S}^2$  with  $F_j$ , a 0-sphere (2 points),  $D_i$  is a disk,  $D_j$  is an interval, and we see that  $D_i \cap D_j$  is a point

**Lemma 4.1.** *If  $D_3$  above is a point then  $F_1$  and  $F_2$  are linked.*

The easiest proof we could find was suggested by Genevieve Walsh. It shows that in this context there is an ambient isotopy of the pair in  $S^5$  that takes the two components to linked great spheres.

*Proof.* If  $X_1$  and  $X_2$  intersect at the origin we already have linked great spheres and are done, so assume they do not intersect at the origin. Let  $S_t^5$  be the round 5-sphere of radius  $t$  centered at the origin in  $R^6$ . Let  $B_t^6$  be the 6-ball bounded by  $S_t^5$  in  $R^6$ . Let  $F_{(i,t)} = (X_i \cap S_t^5)$ . Let  $D_{(i,t)} = (X_i \cap B_t^6)$ . We watch as  $t$  goes from 1 to  $\infty$ .

If we rescale the  $B_t^6$  to unit balls (and thus the  $S_t^5$  to unit spheres and  $F_{(1,t)} \cup F_{(2,t)}$  to a new link in the unit sphere) this yields an isotopy of  $F_1$  and  $F_2$  through  $\hat{S}^5$ . As the isotopy progresses the intersection of the planes containing the spheres moves toward the origin till in the limit it reaches the origin (this is, of course, a direct result of the rescaling - since  $B_t^6$  is shrunk by a factor of  $t$ , all distances, including the distance from  $(D_{(1,t)} \cap D_{(2,t)})$  to the origin, are also shrunk by a factor of  $t$ ). This means that in the limit as  $t \rightarrow \infty$  the link components go to great spheres bounding flat balls in  $\hat{B}^n$  that intersect at the origin and thus the two spheres are just a simple generalization of the Hopf link and are indeed linked. □

Again Figure 11 can help with the intuition of this proof. As we rescale,  $D_i \cap D_j$  moves toward the origin. This implies that in the limit,  $F_i$  becomes a great circle on  $\hat{S}^2$  and  $F_j$  becomes two antipodal points. This confirms what in this simple case we already knew,  $F_i$  and  $F_j$  are linked!

Thus, we have

**Lemma 4.2.** *If  $F_i$  and  $F_j$  are not pairwise linked then  $D_i \cap D_j = \emptyset$ .*

## 5 Brunnian spheres

With the Lemmas in the previous section, now Theorem 2.1 becomes fairly easy to generalize to higher dimensions using the Freedman and Skora's techniques (the convex proof, however, requires completely different techniques).

**Theorem 5.1.** *If  $L$  is a Brunnian link in  $\hat{S}^n$ , it cannot be made up out of only round components.*

The proceeding theorem will follow directly from the following theorem.

**Theorem 5.2.** *Any link in  $\hat{S}^n$  of round spheres with at least three components and all the components pairwise unlinked is the unlink.*

*Proof.* Let  $L = F_1 \cup F_2 \cup \dots \cup F_m$  be any link in  $\hat{S}^n$  with all the components pairwise unlinked, and with each of the  $F_i$  represented by round  $v_i$ -spheres in  $\hat{S}^n$ .

As in Lemma 4.2  $F_i$  bounds a ball  $D_i = X_i \cap \hat{B}^{(n+1)}$ . By Lemma 4.2 we may assume  $D_i \cap D_j = \emptyset$  for all  $i \neq j$ .

Again, let  $S_t^n$  be the round sphere of radius  $t$  centered at the origin in  $R^{(n+1)}$ . Let  $B_t^{(n+1)}$  be the ball bounded by  $S_t^n$  in  $R^{(n+1)}$ . Let  $F_{(i,t)} = (X_i \cap S_t^n)$ . By general position we can assume that none of the  $D_i$  contain the origin.

As we watch  $t$  go to 0, each  $F_{(i,t)}$  will shrink to a point and then disappear when  $D_i$  becomes tangent to  $S_t^n$ . Rescaling the  $S_t^n$  to unit spheres yields an isotopy of the generalized link in  $\hat{S}^n$  that keeps the components disjoint, but shrinks each component to a point showing we have a generalized unlink (after rescaling we consider  $\hat{S}^n$  fixed and let the knots move instead of vice-versa).

□

**Corollary 5.3.** *No  $(m, k)$ -Brunnian link in  $\hat{S}^n$  can be built out of round circles (spheres) if  $k > 2$ .*

*Proof.* Let  $L$  be an  $(m, k)$ -Brunnian link in  $\hat{S}^n$ . Let  $L' \subset L$  be a  $k$  component sublink of  $L$ .  $L'$  is not an unlink by the definition of  $(m, k)$ -Brunnian links in  $\hat{S}^n$ , but also by the definition of  $(m, k)$ -Brunnian links every sublink of  $L'$  is an unlink in  $\hat{S}^n$ . Thus,  $L'$  is a Brunnian link in  $\hat{S}^n$  and cannot be built out of round spheres. Since  $L' \subset L$ ,  $L$  clearly cannot be built out of round spheres either. Note that as with all results in the paper on knots or links in  $\hat{S}^n$  if we set  $n = 3$  the result holds true and shows the theorem holds in the traditional setting of Debrunner's original generalization of Brunnian links.

□

## 6 An example of a Brunnian link in $R^4$

In this section we give an example to show that Brunnian links in higher dimensions do exist. In this case it is easier to do the proof in  $R^n$  than  $\hat{S}^n$ ,

so we will look at a link in  $R^4$ , although again, as with all the theorems in this paper, the result holds in either context.

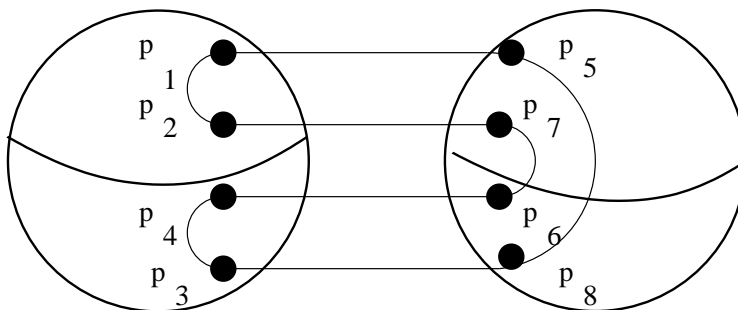


Figure 12: The projection of a Brunnian link in  $R^4$  into  $R^3$  by  $\Pi_\omega$

Let  $(x, y, z, w) \in R^4$ . Let  $\Pi_\omega$  project  $R^4$  onto  $R^3$  via the map  $\Pi_\omega((x, y, z, w)) = (x, y, z)$ .

Let  $L = F_1 \cup F_2 \cup k$ , where  $F_1$  and  $F_2$  are 2-spheres with  $w$  coordinate always equal to 0, and  $k$  is a knot with the property that  $p_i \in k$  has  $w$  coordinate equal to  $-1^i$  for  $i \in \{1, 2, 3, \dots, 8\}$ . Figure 12 shows  $\Pi_\omega(L)$  and note that  $\Pi_\omega(k) \cap \Pi_\omega(F_1) = \Pi_\omega\{p_1, p_2, p_3, p_4\}$  and  $\Pi_\omega(k) \cap \Pi_\omega(F_2) = \Pi_\omega\{p_5, p_6, p_7, p_8\}$ . Let  $s_{(i,j)}$  be the component of  $k - (p_i \cup p_j)$  that intersects  $F_1 \cup F_2$  the fewest times when both are projected into  $R^3$  (so for example,  $\Pi_\omega(s_{(1,2)})$  is contained inside  $\Pi_\omega(F_1)$  except where it intersects it at its end points).

**Theorem 6.1.**  $L$  is a Brunnian link.

*Proof.* Note that if  $k$  is deleted, clearly  $F_1$  and  $F_2$  are not linked, but also if  $F_1$  is deleted,  $k$  can be isotoped in  $R^4$  such that every point of  $s_{(5,7)}$  has  $w = -1$ , keeping the rest of  $k$  fixed, and keeping  $k$  disjoint from  $F_2$  throughout the process. Similarly  $s_{(6,8)}$  can be isotoped to have  $w = 1$ . Now it is clear that these two arcs can be pulled to the inside of  $F_2$  so that  $\Pi_\omega(k)$  is entirely contained inside of  $\Pi_\omega(F_2)$ . Now leaving the  $x, y$ , and  $z$  components fixed, (and thus  $\Pi_\omega(k)$  disjoint from  $\Pi_\omega(F_2)$ )  $k$  can be isotoped so that all the points on  $k$  have  $w = 1$ , and then,  $k$  can clearly be isotoped so that  $\Pi_\omega(k)$  is entirely outside of  $\Pi_\omega(F_2)$ , showing that this is indeed an unlink. The same type of argument can be made for  $F_1$  and  $k$ .

Now that we have shown that every sublink of  $L$  is an unlink we need only show that  $L$  is not the unlink and we will know that it is indeed Brunnian.

This, however, follows quickly from Van Kampen's theorem. Van Kampen's Theorem implies that the the fundamental group of  $R^4 - (F_1 \cup F_2)$  is the free group on two generators and that  $k$  is then a commutator for that group. Thus,  $K$  does not bound a disk in  $R^4 - (F_1 \cup F_2)$  and  $L$  is not an unlink.

□

## 7 Open Questions

We end with a few remaining open questions.

**Question 7.1.** *Is there a Brunnian link other than the Borromean Rings that can be formed out of convex curves?*

The combinatorics get much more complicated as the number of components goes up. Bob Davis proved that the answer is no for  $n = 5$ , but a new strategy seems necessary for large enough  $n$ .

**Question 7.2.** *For a fixed  $n \geq 6$  how many  $n$  component Brunnian links can be formed out of planar curves?*

For each  $n$  the answer is at least one, but perhaps all such examples are from the relatively small family of Brunnian links that are formed by iterated doubles of the Hopf link.

**Question 7.3.** *Is there a Brunnian link in  $\hat{S}^n$  other than the Borromean Rings that can be formed out of convex embeddings of spheres in  $\hat{S}^n$  for any  $n$ ?*

This question appears to be wide open. Perhaps there is a good example of such a link or perhaps there is a proof that no such link exists.

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