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Probabilistic Borda rule voting

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Abstract. An alternative voting system, referred to as probabilistic Borda rule, is developed and analyzed. The winning alternative under this system is chosen by lottery where the weights are determined from each alternative's Borda score relative to all Borda points possible. Advantages of the lottery include the elimination of strategic voting on the set of alternatives under consideration and breaking the tyranny of majority coalitions. Disadvantages include an increased incentive for strategic introduction of new alternatives to alter the lottery weights, and the possible selection of a Condorcet loser. Normative axiomatic properties of the system are also considered. It is shown this system satisfies the axiomatic properties of the standard Borda procedure in a probabilistic fashion.

1 Introduction

Social Choice theorists are often interested in determining which decision processes come closest to generating outcomes that best represent the group's preferences, or a "fair" outcome. One standard way of making group decisions is to rely on a system of majority rule (MR). While this is certainly one of the simplest forms of voting, majority rule suffers from several potential drawbacks, such as tyranny by the majority whereby a stable coalition

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comprised of a minimum majority will always receive their top preference as the outcome and others outside the coalition never do when their preferences differ from the majority's. In addition, when there are more than two alternatives, no alternative may generate a pure majority of votes over all competitors. Most of the normative benefits derived from adopting majority rule decision-making are limited to the case of only two alternatives (May 1957; Rae 1969; Mueller 1989a, b).

The main flaws inherent in simple majority rule have led others to consider alternatives to this standard method of calculating votes, especially for the general case when there might be more than two alternatives from which to choose. One such rule is Borda's rule (BR) which asks voters to rank all the alternatives which are then weighted before aggregation. The higher an individual ranks a particular alternative, the more points that alternative will receive. Specifically, an alternative is awarded one point for each competitor ranked below it in the voter's ranking. The alternative with the most total points is declared the winner. This is one method that may break the simple majority tyranny since the majority coalition's preferences are balanced by the minority's interests. An alternative ranked best by the majority might not win if a large minority ranks that same alternative low and there is another alternative ranked high by both groups. Thus, Borda's rule is thought to generate the fair compromise (Black 1976).

Alternatively, the majority's tyranny can be broken by switching from deterministic majority rule voting where a simple majority automatically and always wins, to a probabilistic lottery mechanism where the probability of any alternative winning is a function of its majority rule votes.[1] Here, the stable majority coalition is likely to win only in proportion to its group size so that over a series of decisions, the minority's interests will also be recognized relative to its size. The actual purpose of the lottery is its use as a threat to entice risk averse voters to offer new compromise alternatives that can be unanimously agreed upon over the risky lottery outcomes. The Pareto compromise is then a function of voter's risk attitudes but Mueller (1989b) argues this may actually be beneficial.

Since the main advantage to probabilistic majority rule (PMR) is its theoretical ability to find a fair compromise for the entire group, it has much in common with the promise of BR voting although the procedure is inherently different. For large group settings under probabilistic majority rule, it is not realistic to expect the unanimous compromise to arise (the transaction costs would be overwhelming except for small committees) and in the case of electing representatives, it may not be viable. Still, the lottery by itself is advantageous in breaking the majority's tyranny.

When there are only two alternatives available, Borda's rule is the same as majority rule (and consequently, plurality rule (PR)). But for more than two

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[1] This type of voting mechanism was first considered by Zeckhauser (1969) to better account for intensity of preferences which is not revealed under standard majority rule.

alternatives, the Borda rule more directly finds the ‘‘compromise’’ alternative, among the existing alternatives, thereby mitigating potentially large transaction costs in searching for new unanimous compromises.[2] However, deterministic Borda voting can still suffer from small minority interests being consistently overwhelmed by the larger majority, even if the large majority is heterogeneous among its non-best alternatives. Thus, for a series of votes, a lottery system which allows groups an opportunity to win in proportion to their size may represent a fair procedure.

The Borda rule is not without its detractors. One of the major criticisms leveled at the Borda rule is its subjectivity to strategic voting, so the true preferences of voters are not revealed (Black 1976). As Borda himself noted, ‘‘My scheme is intended only for honest men’’ (quoted in Mueller 1989a, p. 112). Under this condition, it would not have much practical use. Of course Gibbard (1973) and Satterthwaite (1975) have proven that no (deterministic) vote rule can be designed to be strategy-proof. Still, BR voting is thought to be more prone to strategic voting than many other reasonable rules (Saari 1990).[3] As will be shown below, a further advantage to the lottery mechanism is that it eliminates the incentive to vote strategically since ranking lesser alternatives higher increases the probability the lesser alternatives will win at the expense of more preferred alternatives, and every alternative (except an alternative rated dead-last by everyone) has a nonzero probability of winning.

One final advantage to probabilistic voting schemes is that the size of the vote pluralities matters. Riker (1963) has argued that candidates and parties do not attempt to maximize votes or the probability of victory, rather they are interested only in generating enough votes to win. Thus, his ‘‘size principal’’ indicates the formation of minimum winning coalitions, which are in a position to appropriate larger amounts from others outside the winning coalition the more people that are excluded from the winning side. Under the lottery, increasing the size of the plurality increases the probability of victory, so coalitions should generally be larger and harmful expropriations would be lessened.

Thus, this study considers a new procedure which combines the benefits of deterministic BR voting with the advantages of a fair lottery. In the first stage, normal Borda rankings are developed. Then a lottery is held where each alternative’s chance of winning is equal to its Borda score relative to the total of all Borda points available. One unfortunate consequence is that the incentive to strategically introduce inferior alternatives, which the

[2] The compromise alternative under Borda’ s rule need not be Pareto preferred to the status quo.

[3] However, Saari (1990) has also shown that among positional systems, the Borda rule minimizes the likelihood of successful manipulation by a small percentage of strategic voters, thereby somewhat mitigating this concern. In a world of incomplete information, it is also more difficult to properly vote strategically under Borda’s rule compared to majority rule since a more complete preference profile for the other voters is needed (Mueller 1989a).

deterministic Borda count also suffers from, may actually be enhanced by the lottery under certain conditions as discussed below. This disadvantage to the lottery is largely mitigated, however, when there is imperfect information, and thus in a practical setting, is less cause for concern. Furthermore, the incentive for strategic voting once the list of alternatives is set is completely eliminated through the lottery mechanism, even under perfect information. Finally, it is also shown that a Probabilistic Borda Rule (PBR) satisfies the normative properties of deterministic BR voting developed by Young (1974), and some additional properties are also discussed.

2 The rationale behind Borda's rule

As with any voting procedure, BR is subject to several potential criticisms. The primary objection by many opponents of BR entails the likelihood of strategic behavior under this system of preference aggregation. The extra information this rule requires (a complete preference ordering as opposed to plurality or majority rule where only the top choice is needed) allows more room for strategic behavior by voters, either by altering their revealed rankings or by introducing new irrelevant alternatives simply to affect the order of the original alternatives in the group aggregation, even when this new alternative is ranked very low by most voters (Ordeshook 1986).[4] This extra information requirement, however, also limits voters' strategic behavior when information is incomplete since voters would not be able to develop a better strategy to sincere voting unless they are able to determine the Borda scores prior to voting which requires information on all other voters' complete preference profiles (Mueller 1989a).

The Borda rule may also fail to choose the Condorcet winner (the unique winner of all pairwise comparisons) if and when one exists. However, among a large class of alternative vote rules, the Borda count has been shown to be most likely to choose the Condorcet winner when the number of alternatives is large (Mueller 1989a; Van Newenhizen 1990). Furthermore, strong advocates of BR argue that the potential dichotomy between the Condorcet and Borda winners is a flaw not inherent in BR, but rather to Condorcet's procedure of relying strictly on pairwise comparisons since such a procedure ignores the importance of transitivity inherent in individual preference profiles (Saari 1994, 2000). Borda's rule is normally depicted as a positional system which implies the information contained in individual transitivity is retained, yet this rule can also be envisioned as a pairwise voting system (Levin and Nalebuff 1995; Young 1997). However, it does not fall prey to the limited information contained in Condorcet's pairwise rule because BR also considers the size of the pairwise vote differentials.

[4] Dummett (1998) has suggested a slight complication to the Borda procedure to limit the incentive for the strategic introduction of new alternatives.

In addition, Saari (1994) shows that BR "is the unique method to minimize the number and kinds of paradoxes, to minimize the likelihood of a paradox, to minimize the likelihood that a small group can successfully manipulate the outcome, to minimize the possibility of voters' errors adversely changing the outcome, and so forth" (p. 14). From an axiomatic standpoint, Young (1974) shows the Borda procedure is the unique procedure which satisfies certain attractive normative properties.[5] These properties are described in a later section, along with some additional BR properties, where it is also shown which properties are still satisfied in a probabilistic sense under the lottery mechanism.

For most proponents of Borda's rule, however, the primary advantage to this procedure is its ability to find the "fair" compromise since it includes more information from the voters than either plurality or majority rule (Black 1976; Dummett 1984; Saari 1990, 1994). Still, small minorities may be consistently left out, in which case it is not clear that the winner under BR is truly the "fair" outcome. Indeed, no "fair" outcome may exist from different group perspectives. If it also deemed desirable for the preferences of minority groups to be counted in relation to their numbers, then BR might not go far enough. The lottery mechanism refines the weighting scheme under BR so that although the BR winner has the best chance of winning, it will not necessarily win every time. More importantly, although as Mueller (1989a) and Saari (1990) demonstrate the potential for strategic voting on a given set of alternatives is not as great as typically assumed, BR is still not immune to this problem. However, it will be shown below that the addition of the lottery ensures sincere preference voting on the available set of alternatives.

3 The probabilistic Borda rule procedure

Consider a set of voters, $N = (1, ..., n)$ deciding among a set of $Z = (1, ..., z)$ alternatives, where each voter x has a preference order, D_x, consisting of a strong order of all the possible alternatives. The profile $D = (D_1, ..., D_n)$ is the vector of orderings of D_x on Z. Define the total number of points assigned to any alternative j as Q_j.

Borda's rule assigns points to each alternative for each voter such that a voter's j'th preference is assigned $z - j$ points, for all $j = \{1, ...z\}$. The group choice is then the option with the greatest number of total points. The PBR method aggregates votes the same as in BR, but a second-stage is added whereby each alternative j is assigned a probability π_j based on its aggregated

[5] The axiomatic approach is more useful to characterize a certain procedure rather than to choose among alternative procedures. Any reasonable vote rule can be shown to be unique in satisfying certain specific seemingly necessary normative properties. Even the much properly-maligned plurality rule uniquely satisfies a specific set of normative properties (see Ching 1996).

Borda score relative to all the others as determined by the total of all points, i.e. $\pi_j = Q_j/\Sigma Q_i$, where $\Sigma Q_i = n(z-1+z-2+\cdots+1)$. A winner is then chosen using a random process (such as a random number generator or drawing balls from an urn) based on these weighted probabilities.

Thus, there are two main distinctions between the deterministic and probabilistic versions of the Borda Rule. Under PBR the alternative with a plurality of points is not necessarily chosen as it would be under BR, although it has the greatest likelihood of being selected in the lottery. However, since the lottery weights depend on the Borda scores relative to each other, the size of the plurality differences is also important, thereby yielding further incentive for candidates to attempt to appeal to a broader base of voters.

Once the lottery is enacted, any alternative which received at least a single point may be possibly selected. This implies, as with all lottery mechanisms, that the Condorcet winner (CW), if it exists, may not be chosen, and also that a Condorcet loser (CL), if it exists, may be chosen. This potential drawback to lottery systems can also be found in many standard deterministic procedures. Specifically, any positional method (including the most commonly utilized plurality rule) may fail to select the CW and, with the exception of BR, may also potentially select the CL (Saari, 1994). Since BR will always assign more points to a CW than a CL, if they both exist, the lottery is more likely to select the CW than the CL. Specifically, since under BR, $Q_{CW} > \Sigma Q_i/z$ and $Q_{CL} < \Sigma Q_i/z$, then under the lottery, $\pi_{CW} > 1/z$ and $\pi_{CL} < 1/z$, suggesting the CW has a better than even chance of being selected, and the CL has a less than even chance of being selected. Thus, in the lottery CW is always a favored, but not necessarily the favorite, alternative to emerge victorious, (the favorite would always be the BR winner) and CL is always a disfavored alternative, but not necessarily the least likely to win.[6]

4 Strategic voting under the lottery

It can be easily shown that knowledge of the second-stage lottery eliminates the incentive to vote strategically in the first round. Mueller (1989b) discusses the case of a single round of plurality voting before the lottery, and his argument is extended here for Borda weighting. Define $U_x(i)$ as the utility voter x receives from alternative i. Voter x's expected utility under the PBR is then $EUx = \Sigma[\pi_i U_x(i)]$for $i = 1, ..., z$. Strategic voting entails some manner of ranking less preferred alternatives above more preferred alternatives.

[6] I thank Don Saari for suggesting this result. The Borda vote representations are based on Fishburn and Gehrlein's (1976) result that a Condorect winner will always receive more than the average number of Borda points possible, and a Condorcet loser will always receive less than the average number of Borda points possible. Thus follows the conclusion that a Condorcet winner will always be ranked above a Condorcet loser under BR. See also Saari (2000) for additional proofs.

Consider any particular set of alternatives i and j such that $U_x(i) > U_x(j)$. Voter x votes strategically by submitting a substitute preference order D_x' listing j above i which then results in new lottery probabilities such that $\pi_j' > \pi_j$ and $\pi_i' < \pi_i$ but $\pi_j' + \pi_i' = \pi_j + \pi_i$. As Zeckhauser (1973) notes, "The key element here is that when the individual switches his vote the probability gain for the switched-up candidate is just equal to the probability loss for the switched-down candidate" (p.938). Since strategic voting will only change the probabilities of an alternative winning but not the utility the voter receives from any given alternative, then it must be the case that $EU_x' < EU_x$, which means the voter's expected utility is maximized by sincere voting. This argument can easily be extended for strategic voting involving the switching of any number of alternatives. (See Zeckhauser 1973.)

A lottery mechanism following any single round first-stage rule will eliminate the incentive to vote strategically in the first round (Mueller 1989b). Strategic voting occurs under deterministic vote rules when voters think their best alternative cannot win if they vote sincerely. Under probabilistic Borda voting, however, every alternative not rated last by everyone has a non-zero probability of winning so non-sincere voting will simply lower the probability of a non-zero probability alternative in favor of a less preferred alternative by an equal amount thereby reducing expected utility.[7]

Not all lottery-based schemes, however, are strategy-proof. For example, Probabilistic Majority Rule on more than two alternatives, whereby each pair of alternatives is considered in turn over a series of rounds in order to develop the lottery weights, can still lead to strategic voting (Mueller 1989b). It is only when all alternatives are voted on simultaneously as in a pure plurality or Borda format, that the incentive to vote strategically is eliminated.

5 Strategic introduction of new alternatives under the lottery

Although the incentive to vote strategically in the first round is eliminated, the incentive to strategically offer new alternatives is still present. Consider the following example of perfect information in Table 1 where each voter's preference is ranked from top to bottom on alternatives a, b, c.

Under BR, points are allocated such that $Q_a = 8$, $Q_b = 7$, $Q_c = 6$ and a is the chosen winner with probability 1. Under PBR, $\pi_a = 8/21$, $\pi_b = 7/21$, $\pi_c = 6/21$ and a is most likely to win, but the winning outcome is yet to be determined. Under BR, Voter 7 has the incentive to offer a new alternative d such that the preference orders are changed as in Table 2.

The outcomes now are $Q_a' = 11$, $Q_b' = 12$, $Q_c' = 13$, $Q_d' = 6$ and $\pi_a' = 11/42$, $\pi_b' = 12/42$, $\pi_c' = 13/42$, $\pi_d' = 6/42$. The new alternative d finishes dead

[7] This also explains why a probabilistic plurality vote will eliminate strategic voting since anyone's most preferred alternative has a strictly positive probability of winning under the lottery.

Table 1. Voter preferences over three alternatives

Voter 1	Voter 2	Voter 3	Voter 4	Voter 5	Voter 6	Voter 7
a	*a*	*a*	*b*	*b*	*c*	*c*
b	*b*	*b*	*c*	*c*	*a*	*a*
c	*c*	*c*	*a*	*a*	*b*	b

Table 2. Voter preferences under strategic introduction of new alternative

Voter 1	Voter 2	Voter 3	Voter 4	Voter 5	Voter 6	Voter 7
a	*a*	*a*	*b*	*b*	*c*	*c*
b	*b*	*b*	*c*	*c*	*d*	*d*
c	*c*	*c*	*d*	*d*	*a*	*a*
d	*d*	*d*	*a*	*a*	*b*	*b*

last under BR. Even though this alternative cannot win under BR, its presence has reversed the order of the original alternatives aggregate ranking, making *c* the new winner.[8] This well-known inverted order paradox is due to the Borda rule's violation of the Independence of Irrelevant Alternatives axiom (Ordeshook 1986).[9] Under PBR, *c* is not definitively the winner, but its chances have greatly improved. However, alternative *b*'s chance of winning, relative to *a*, has also increased which is worse for Voter 7. Thus, although this strategic introduction alternative is definitely better for Voter 7 under deterministic outcomes, its appeal under the lottery depends on the change in relative probabilities vis-a-vis the relative utility each alternative will bring. In other words, if *c* is only slightly preferred to *a* but *a* is greatly preferred to *b*, the introduction of *d* might reduce Voter 7's expected utility.

The difference between strategic voting over a given set of alternatives, discussed in the previous section, and strategically introducing new alternatives is that the former only affects $Q_j^{'}(\pi_j^{'})$ for certain (or all) $j \epsilon Z$, whereas the latter also affects $\Sigma Q_i^{'}$ by increasing z to $z^{'} = z + 1$. The introduction of a new alternative can therefore raise the probability of an individual's most preferred alternative winning relative to other alternatives, for certain sets of preference profiles. Once the choice of alternatives is set, however, as shown above it is always the best strategy to vote sincerely prior to the lottery.

[8] The "legitimacy" of the BR winner is called into question *if d* was strategically introduced specifically to make *c* the winner. Notice however, a majority of voters prefer this new outcome. If *d* were one of the original alternatives such that Table 2 represents the starting point, the problem would simply work in reverse in that Voter 1 would have an incentive to find a way to disqualify *d* for the sole purpose of making *a* the new winner (as in Table 1). Then a majority of voters would be hurt by the new outcome.

[9] Chamberline and Courant (1983) report on a study suggesting that in practice BR is less susceptible to the inverted order paradox than other common voting rules.

Table 3. Example of voter preferences on two alternatives

Voter 1	Voter 2	Voter 3
a	*b*	*b*
b	*a*	*a*

Table 4. Strategic introduction of unanimously worst alternative

Voter 1	Voter 2	Voter 3
a	*b*	*b*
b	*a*	*a*
c	*c*	*c*

In other cases, PBR may be more prone, and easier to manipulate in terms of introducing new alternatives prior to voting. Consider the following simple example in Table 3 for 3 voters and only two alternatives.

Since there are only two alternatives, BR, MR and PR are equivalent, yielding $Q_a = 1$, $Q_b = 2$. Likewise, the lottery mechanism under any of these rules yields $\pi_a = 1/3$, $\pi_b = 2/3$. Voter 1 has the incentive under PBR to introduce a new alternative that everyone unanimously agrees is worse than either of the two original alternatives, as shown below in Table 4.

Although the winner is not affected by the introduction of this new alternative under deterministic voting (the ranking is unaltered), the exact Borda total is altered which is enough to make Voter 1 better off under the Borda lottery. With the addition of a new bottom alternative, each of the original alternative's vote total is raised by n. Thus under PBR, the probabilities are transformed as $\pi_a^{'} = 4/9$, $\pi_b^{'} = 5/9$, $\pi_c^{'} = 0$. There is no risk from the new bottom alternative winning, and the chance of Voter 1's preferred alternative being selected is increased.

In general, the introduction of a new unanimously worst alternative will transform the probabilities for an original alternative j as $\pi_j^{'} = (Q_j + n)/(\Sigma Q_i + zn)$ so that the change in lottery probability $\pi_j^{'} - \pi_j = n(\Sigma Q_i - Q_j)/\Sigma Q_i(\Sigma Q_i + zn)$. Since the only factor unique to alternative j is $-Q_j$ in the numerator, the increase in probability is inverse to Q_j so the previously lowest collectively ranked alternative will always benefit the most from the introduction of a new unanimously worst alternative. Thus, any voter whose preference is the exact inverse of the original Borda ranking will benefit from the introduction of such an alternative. This does not however lead to an endless cycle of new introductions. Once a unanimously worst alternative is introduced, no voter can now have an inverse preference ranking to the Borda score ranking, although strategic introduction can still occur depending on, as outlined above following the example set forth in Tables 1 and 2, the change in probability differentials of various alternatives relative to their utility differentials. Again, when the time comes to actually vote on whatever is the final set of alternatives, voting sincerely will be a dominant strategy for all voters.

6 Axiomatic properties of PBR

In a well-known study, May (1952) has shown that majority rule on a binary choice set is the only rule which satisfies the four specific properties of decisiveness, anonymity, neutrality, and positive responsiveness. Mueller (1989b) extended May's work by showing that Probabilistic Majority Rule (MR with a lottery in the second stage) satisfied three of the conditions in a probabilistic manner.[10] Since the BR simplifies to MR in the case of two alternatives, Mueller's results hold for PBR as well. It is now shown how the Probabilistic Borda Rule relates to the axiomatic properties for the traditional (deterministic) Borda rule developed by Young (1974) for any number of alternatives z. Young proved that Borda was the only rule which simultaneously satisfies the properties of neutrality, cancellation, faithfulness, and consistency, and properly defines a choice set of "best" elements for any set of profiles D.

Choice set: A choice set $C(D, Z)$ contains only those alternatives j for which $Q_j \geq Q_i$ for all $i \epsilon Z$.

Neutrality: $C(h[D], Z) = h[C(D, Z)]$, where the permutation on alternative names, h, leads to the same permutation on the choice set.

Cancellation: Define $\Gamma_{ij}(D)$ to be the number of voters who prefer alternative i to alternative j. If $\Gamma_{ij}(D) = \Gamma_{ji}(D)$ for all i, j then under cancellation $C(D, Z) = \{1, ..., z\}$.

Faithfullness: If $n = 1$, then $C(D, Z) = \{j\}$ if and only if $U_n(j) > U_n(i)$ for all $i \neq j$.

Consistency: Let N^* and N^{**} be subsets of N such that $N = N^* \cup N^{**}$ and $N^* \cap N^{**} = \emptyset$, and D^* and D^{**} are the profile vectors on N^* and N^{**}. Consistency requires that if $C(D^*, Z) \cap C(D^{**}, Z) \neq \emptyset$, then $C(D, Z) = C(D^*, Z) \cap C(D^{**}, Z)$.

A choice set is a list of alternatives that are at least as good as everything else in the sense that it is ranked at least as high by the aggregation procedure as all the other alternatives. The neutrality criteria, similar to May's definition for two alternatives, prescribes that no single alternative is favored by the voting procedure over any of the others. The switching of any (or all) alternatives by every voter leads to a duplicate switching of these alternatives in the aggregated group ranking. Cancellation is similar to May's anonymity

[10] The exception being that PMR is not decisive in that repetitions of the voting can result in different winners chosen under the lottery (assuming the absence of unanimous consent on the preferred alternative), although the lottery weights will remain unaffected each time the vote is recast. Ching (1996) develops the normative properties of plurality rule for any number of potential alternatives. I am unaware of any study that has considered these properties in a PR lottery framework in the manner Mueller (1989b) has done for PMR.

principle where the identity of individual voters does not affect the outcome. Any one voter can cancel out the preferences of any other individual voter with diametrically opposed preferences. Specifically, the condition requires that if the number of voters preferring one specific alternative to another is equal to the number who prefer the other alternative, and this is true for all pairs of alternatives, then the choice set must contain all the potential alternatives. Faithfullness requires that when there is only one individual in the group the choice set consists only of the top preference of that individual. Finally, consistency requires that the combination of two groups yields the choices the separate groups held in common. Note that if the vote procedure is both faithful and consistent, then it is also Pareto, i.e., if one alternative is most preferred by everyone, than it will be the only alternative in the choice set.

PBR satisfies probabilistic versions of the above properties, except that it does not properly define a choice set. The procedure is neutral among the alternatives since a permutation on alternative names leads to a corresponding permutation on the probabilities of each alternative winning. Similarly it satisfies a version of cancellation in that if for all pairs of alternatives the number of voters preferring the first alternative is equal to the number preferring the second alternative, then the Borda count will be equal for all alternatives and the probabilities for all alternatives winning under the PBR will be identical ($1/z$). In addition, PBR is probabilistically faithful in that when there is only one person in the group, that person's top choice has the greatest chance of winning. Finally, the rule is probabilistically consistent as the alternatives in common ranked highest by different subgroups will also be ranked highest by the collective and therefore have the highest probability of winning.

In the strict sense, any vote procedure that contains a random element does not properly define a choice set. For any single vote, the winning alternative under PBR is most likely to be the one with the highest vote total, but repeated voting can lead to any alternative not unanimously ranked last being chosen even when preferences remain stable. This is the main differential between the probabilistic and deterministic versions of Borda's rule. Anyone with the authority to call a vote may be so inclined to repeat the same exercise with the same alternatives if that person feels he faired poorly under the most recent lottery. In a deterministic setting, repetition would always lead to the same outcome when preferences remain stable.

Now reconsider the faithfulness property. Although PBR is probabilistically faithful, that it does not directly satisfy the condition of faithfulness may be cause for concern. Even an individual making a decision for herself is subject to the lottery and may not get her wish. More generally, this reveals that the procedure is not Pareto; unanimous consent on a best alternative may not be respected under the lottery (unless there are only two alternatives), although it is probabilistically Pareto in the sense that a unanimously preferred alternative generates a greater probability of being selected than any other specific alternative. The probabilistic majority rule procedure described by Mueller (1989b) allows unanimous consent (on a new alternative) to avoid the lottery. A similar modification can be introduced to PBR on the existing alternatives.

Under this modification, designate alternative j the automatic winner if everyone ranks it first, that is if $Q_j = n(z-1)$ for any j. Otherwise the lottery is held based on the Borda vote totals as in PBR. The potential downside is this rule is no longer strategy-proof. If there is one person in the group that does not rank j first, but everyone else does, then the lottery is held over all alternatives. It may therefore be in that person's interest to strategically rank this alternative first to avoid the risky lottery. Notice, however, that under this scenario everyone else also prefers this individual to vote strategically.

The concern over the failure to respect Pareto may still exist even if there is no single alternative unanimously ranked best by everyone. Suppose there is a subset of alternatives $S\epsilon Z$ such that any alternative in S is unanimously preferred over any alternative not in S. It would be unanimously agreed upon that the winner should be limited to only those alternatives in S. The way to ensure this is to compute the Borda rankings only on the subset S. This modification will eliminate the strategic introduction of a unanimously worst alternative (as in the example in Table 4) unless that person is also willing to take the risk of strategically placing the new alternative higher in her ordering to ensure it is included in the lottery (in which case S is an empty set). Again, this is a very risky strategy but may be worthwhile under certain sets of preference profiles.

There are a considerable number of additional properties that BR satisfies. Most of the properties scholars develop compare the specific alternative(s) chosen under differening sets of alternatives, preferences, or voters. Whenever the BR winner switches from one alternative to another, the PBR favored alternative switches in the same manner, as in a probabilistic version of neutrality. Whenever the BR winner remains the same, the PBR favored alternative remains the same as well, as in a probabilistic version of consistency. The exact probabilities, however, may be altered.

Consider for example the property of monotonicity, which requires that if $a\epsilon Z$ is chosen the winner, then a still wins when the only change is that at least one voter places a higher in her ranking. I.e., if $C(D,Z) = a$, then $C(D',Z) = a$ when the only difference between D and D' is as just described. Borda's rule, as true for any positional method, satisfies monotonicity. PBR, therefore, satisfies monotonicity probabilistically in that under both scenarios, a is the favored alternative. Furthermore, since $Q'_a > Q_a$,and $\Sigma Q_i = \Sigma Q'_i$,then $\pi'_a > \pi_a$ which means a is favored to a greater degree under D'. Note this is only possible if at least one other alternative's probability of being selected in the lottery is reduced. This suggests an additional degree of responsiveness under the lottery. The property of monotonicity only applies to the winning alternative, and does not require any specific relationship to hold among the remaining alternatives. Under PBR, every alternative that is raised in a voter's ranking is rewarded with a higher probability of being selected in the lottery, and every alternative that is lowered in a voter's ranking is penalized with a lower probability of being selected in the lottery.

This can be contrasted with a lottery on plurality votes. Plurality is also a positional method and satisfies monotonicity, and likewise the lottery satisfies monotonicity probabilistically as described above for PBR. However, the responsiveness to such a change would only alter the lottery probabilities from a plurality vote if a voter alters her top ranking. An alternative is not rewarded with a greater plurality probability for being ranked higher, unless it is now top ranked, and no alternative is penalized under the lottery for falling lower in a voter's ranking unless it had previously been top ranked. Thus a lottery under Borda is more responsive to voter profile changes than a lottery based strictly on plurality rule votes.

7 Conclusions

Democracy is sometimes equated with majority rule, but scholars have long noted the problems inherent in a majority rule system. One of the most harmful aspects of majority rule is its theoretical possibility to lead to a tyranny by a bare majority over the (potentially large) minority. Borda's rule reduces but does not eliminate the possibility for large stable coalitions to dominate voting outcomes. The lottery mechanism proposed here ensures that every voter's full ranking will contribute to the possibility that any particular alternative is chosen, which in some ways may be considered more democratic.

Under certain sets of voter preference profiles, the probabilistic Borda rule may suffer from the incentive for individuals to strategically offer new alternatives in order to alter the lottery probabilities, not because they genuinely favor the new alternative. This condition, however, depends on the strategic person's ability to accurately estimate the full preference profile of all other voters, which is highly unlikely. Furthermore, in some cases such as mass elections, the voters are not able to introduce new alternatives and for any given set of alternatives, the lottery mechanism ensures voting will be sincere thereby eliminating a serious shortcoming to all deterministic voting rules.

To the degree that Borda's rule is an improvement over Simple Majority Rule for multiple alternatives, and Probabilistic Majority Rule is an improvement over Simple Majority Rule, the Probabilistic Borda Rule may represent a further improvement, by incorporating the best elements of both the Borda count by taking advantage of each voter's full preference order, and PMR by breaking the potential tyranny by stable majority coalitions.

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