Physics 742 – Graduate Quantum Mechanics 2

Midterm Exam, Spring 2021

Please note that some possibly helpful formulas are listed on the next page. Each question is worth twenty points. The points for individual parts are marked in []'s.

1. A spin ½ particle has state operator given by $\rho = N\begin{pmatrix} 2 & 1+i \\ c & 2 \end{pmatrix}$, where N and c are constants.
   (a) [5] What is the normalization constant N? What is the complex number c?
   (b) [15] What are the expectation values of each of the three spin operators **S** (given below)?

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2. A particle of mass m in two dimensions lies in a potential $V = -V_0 e^{-\beta\rho^2}$. Estimate the energy of the ground state using the trial wave function $\psi = e^{-\alpha\rho^2/2}$.

3. In the basis $\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle\}$, the Hamiltonian is given by $H = \begin{pmatrix} A & i\delta & 0 \\ -i\delta & 2A & \delta \\ 0 & \delta & 2A \end{pmatrix}$, with δ small.
   (a) [2] What are the eigenstates and eigenenergies in the limit $\delta = 0$?
   (b) [8] For the ground state, what is the eigenstate to first order in δ and the energy to second order in δ?
   (c) [10] For the first excited states, what are the eigenstates to leading order in δ and the energy to first order in δ?

4. A particle of mass m lies in the 1D potential $V(x) = \begin{cases} \frac{1}{2}m\omega^2x^2 - A & x > 0, \\ \frac{1}{2}m\omega^2x^2 & x < 0. \end{cases}$
   Use the WKB approximation to estimate the energy of the n'th state (assume $E_n > 0$).

5. Imagine that a proton in a hydrogen atom consists of a point charge at the origin of magnitude $\frac{1}{2}e$ and a spherical shell of magnitude $\frac{1}{2}e$ at radius R.
   (a) [12] Find the electric field for $r < R$ (the electric field from a point charge q is $E = k_e q r^{-2}$). Integrate it to find the electric potential $U(r) = -\int E\,dr$ in the interior region. Choose the constant of integration so $U(R) = k_e e R^{-1}$, as it must.
   (b) [3] Find the perturbation due to finite size, $W(r) = -eU(r) - (-k_e e^2 r^{-1})$.
   (c) [5] Find the shift in energy due to the finite proton size, assuming that in the nuclear region $\psi(\mathbf{r}) \approx \psi(0)$. You may leave your answers in terms of $\psi(0)$.

Calculus in 2D:

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\varphi}}, \quad \nabla^2\psi = \frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}, \quad \int f(\rho,\phi)\,d^2\mathbf{r} = \int_0^{2\pi} d\phi \int_0^\infty f(\rho,\phi)\,\rho\,d\rho$$

Possibly Helpful Integrals:

$$\int_0^\infty \rho^n e^{-A\rho^2}\,dx = \tfrac{1}{2}A^{-(n+1)/2}\Gamma\left(\tfrac{n+1}{2}\right), \quad \Gamma(1) = \Gamma(2) = 1, \quad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\sqrt{\pi}, \quad \Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{4}\sqrt{\pi}.$$

$$\int_0^y \sqrt{a-bx}\,dx = \frac{2}{3b}\left[a^{3/2} - (a-by)^{3/2}\right], \quad \int_0^y \sqrt{a-bx^2}\,dx = \frac{a}{2\sqrt{b}}\sin^{-1}\left(y\sqrt{b/a}\right) + \frac{y}{2}\sqrt{a-by^2}.$$