Physics 742 – Graduate Quantum Mechanics 2

Solutions to Last Exam, Spring 2021

Please note that some possibly helpful formulas are listed on the next page. Each question is worth twenty points. The points for individual parts are marked in []'s.

1. A particle with mass μ and momentum $\hbar k$ moving in the *z*-direction scatters from a weak potential $V(x,y,z) = V_0 e^{-x^2/\alpha} e^{-y^2/\alpha} \delta(z)$. Find the differential and total cross-section using the first Born approximation. For the total cross section, you may leave one integral unfinished.

To use the Born approximation, we first need to find the Fourier transform of the potential, which is

$$\int V(\mathbf{r}) e^{-i\mathbf{K}\cdot\mathbf{r}} d^3\mathbf{r} = V_0 \int_{-\infty}^{\infty} e^{-iK_x x - x^2/\alpha} dx \int_{-\infty}^{\infty} e^{-iK_y y - y^2/\alpha} dy \int_{-\infty}^{\infty} \delta(z) e^{-iK_z z} dz$$

$$= V_0 \sqrt{\pi\alpha} e^{\alpha(iK_x)^2/4} \sqrt{2\pi\alpha} e^{\alpha(iK_y)^2/4} = \pi V_0 \alpha e^{-\alpha K_x^2/4 - \alpha K_y^2/4}.$$

We then substitute this into the formula

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2}{4\pi^2\hbar^4} \left| \int d^3\mathbf{r} V(\mathbf{r}) e^{-i\mathbf{K}\cdot\mathbf{r}} \right|^2 = \frac{\mu^2}{4\pi^2\hbar^4} (\pi V_0 \alpha)^2 e^{-\alpha(K_x^2 + K_y^2)/2} = \frac{\mu^2 V_0^2 \alpha^2}{4\hbar^4} e^{-\alpha(K_x^2+K_y^2)/2}$$

Now, $\mathbf{K} = \mathbf{k}' - \mathbf{k} = k(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta - 1)$, and therefore

$$K_x^2 + K_y^2 = k^2 \left(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi\right) = k^2 \sin^2\theta$$

We therefore have

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2 V_0^2 \alpha^2}{4\hbar^4} \exp\left(-\tfrac{1}{2}\alpha k^2 \sin^2\theta\right).$$

The total cross-section is found by integrating over angles:

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{\mu^2 V_0^2 \alpha^2}{4\hbar^4} \int_0^{2\pi} d\phi \int_{-1}^{1} \exp\left(-\tfrac{1}{2}\alpha k^2 \sin^2\theta\right) d(\cos\theta)$$

$$= \frac{\pi\mu^2 V_0^2 \alpha^2}{2\hbar^4} \int_{-1}^{1} \exp\left(-\tfrac{1}{2}\alpha k^2 \sin^2\theta\right) d(\cos\theta).$$

The final integral can be written in terms of error functions (with complex arguments), but that doesn't give any particular insights.

2. A particle of mass *m* lies in a potential $V(x)=\begin{cases}\frac{1}{2}m\omega^2x^2 & x>0,\\ \infty & x<0.\end{cases}$ **It is in the ground state, whose wave function is given in the allowed region (*x* > 0) by** $\psi_g(x)=\sqrt{2}\phi_1(x)$**, where** $\phi_1(x)$ **is the first excited state of the *standard* harmonic oscillator (with** $V(x)=\frac{1}{2}m\omega^2x^2$**). The barrier forcing *x* > 0 is then moved towards minus infinity, so that the potential becomes the standard harmonic oscillator. Find the probability of it ending up in the ground state or first excited state if the barrier is moved (a) adiabatically, or (b) suddenly.**

In the adiabatic limit, the ground state goes to the ground state, so the probabilities are given by $P\left(\left|\psi_g\right\rangle\to\left|0\right\rangle\right)=1$ and $P\left(\left|\psi_g\right\rangle\to\left|1\right\rangle\right)=0$. In the sudden approximation, the probabilities are given by $P\left(\left|\psi_g\right\rangle\to\left|n\right\rangle\right)=\left|\left\langle n\middle|\psi_g\right\rangle\right|^2$, so we have to find the corresponding matrix elements:

$$\left\langle 0\middle|\psi_g\right\rangle=\sqrt{2}\int_0^\infty\phi_0^*(x)\phi_1(x)\,dx=\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\sqrt{2\alpha}\int_0^\infty e^{-\alpha x^2/2}xe^{-\alpha x^2/2}dx=\frac{2\alpha}{\sqrt{\pi}}\int_0^\infty xe^{-\alpha x^2}dx$$

$$=\frac{2\alpha}{\sqrt{\pi}}\frac{1}{2}\alpha^{-1}\Gamma(1)=\frac{1}{\sqrt{\pi}},$$

$$\left\langle 0\middle|\psi_g\right\rangle=\sqrt{2}\int_0^\infty\phi_1^*(x)\phi_1(x)\,dx=\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\left(\sqrt{2\alpha}\right)^2\int_0^\infty\left(xe^{-\alpha x^2/2}\right)^2dx=\sqrt{\frac{8\alpha^3}{\pi}}\int_0^\infty x^2e^{-\alpha x^2}dx$$

$$=\sqrt{\frac{8\alpha^3}{\pi}}\frac{1}{2}\alpha^{-3/2}\Gamma\left(\tfrac{3}{2}\right)=\sqrt{\frac{8}{\pi}}\frac{1}{2}\cdot\frac{\sqrt{\pi}}{2}=\frac{1}{\sqrt{2}}.$$

The probabilities are the squares of these. We therefore have

$$\text{adiabatic:}\quad P\left(\left|\psi_g\right\rangle\to\left|0\right\rangle\right)=1,\quad P\left(\left|\psi_g\right\rangle\to\left|1\right\rangle\right)=0.$$

$$\text{sudden:}\quad P\left(\left|\psi_g\right\rangle\to\left|0\right\rangle\right)=\frac{1}{\pi},\quad P\left(\left|\psi_g\right\rangle\to\left|1\right\rangle\right)=\frac{1}{2}.$$

3. A particle of mass *m* is in the ground state $\left|0\right\rangle$ at $t \to -\infty$ of a harmonic oscillator with potential $V(x) = \tfrac{1}{2}m\omega^2 x^2$, but there is also a perturbation of the form $W(x,t) = Bx^2 e^{-At^2}$, where *B* is small. To first order in perturbation theory, what *other* final states $\left|n\right\rangle$ can it end up in, and what is the corresponding probability in the limit $t \to +\infty$?

To first order, the only non-zero matrix elements are those that are connected by the operator *W*, so we need to look at matrix elements of the form $Be^{-At^2}\left\langle F\right|X^2\left|I\right\rangle$. We see that

$$\left\langle F\right|X^2\left|I\right\rangle = \frac{\hbar}{2m\omega}\left\langle F\right|\left(a+a^\dagger\right)^2\left|0\right\rangle = \frac{\hbar}{2m\omega}\left\langle F\right|\left(a+a^\dagger\right)\left|1\right\rangle = \frac{\hbar}{2m\omega}\left\langle F\right|\left(\left|0\right\rangle + \sqrt{2}\left|2\right\rangle\right)$$
$$= \frac{\hbar}{2m\omega}\left(\delta_{F0} + \sqrt{2}\delta_{F2}\right).$$

We are only interested in states different from the initial state, so we just want $F = 2$. For this state we have

$$\omega_{FI} = \left(E_F - E_I\right)/\hbar = \left(E_2 - E_0\right)/\hbar = \left(\tfrac{5}{2}\hbar\omega - \tfrac{1}{2}\hbar\omega\right)/\hbar = 2\omega.$$

We now calculate the integral

$$S_{20} = \frac{1}{i\hbar}\int_{-\infty}^{\infty} W_{20}(t)\,e^{i\omega_{20}t}\,dt = \frac{1}{i\hbar}B\int_{-\infty}^{\infty}\left\langle 2\right|X^2\left|0\right\rangle e^{-At^2}e^{i\omega_{20}t}\,dt = \frac{1}{i\hbar}\frac{\hbar\sqrt{2}}{2m\omega}B\int_{-\infty}^{\infty} e^{-At^2+2i\omega t}\,dt$$
$$= \frac{B}{i\sqrt{2}m\omega}\sqrt{\frac{\pi}{A}}e^{(2i\omega)^2/4A} = \frac{B}{im\omega}\sqrt{\frac{\pi}{2A}}e^{-\omega^2/A}.$$

We then square the magnitude to get the probability, so

$$P(0\to 2) = \frac{\pi B^2}{2Am^2\omega^2}e^{-2\omega^2/A}.$$

4. A system lies in a superposition of 1 or 2 photons, $|\psi\rangle = N\left(|1,\mathbf{q},\tau\rangle + i\sqrt{2}\,|2,\mathbf{q},\tau\rangle\right)$, where $\mathbf{q} = q\hat{\mathbf{x}}$ and $\boldsymbol{\varepsilon}_\tau = \hat{\mathbf{y}}$.

(a) [3] What is the correct normalization constant *N*?

The state must be normalized, so that

$$1 = \langle\psi|\psi\rangle = N^2\left(\langle 1,\mathbf{q},\tau| - i\sqrt{2}\langle 2,\mathbf{q},\tau|\right)\left(|1,\mathbf{q},\tau\rangle + i\sqrt{2}\,|2,\mathbf{q},\tau\rangle\right) = N^2(1+2) = 3N^2 .$$

We therefore have $N = \frac{1}{\sqrt{3}}$.

(b) [17] What is the expectation value of the electric field for this state?

We simply substitute into our expression for the electric field, so we have

$$\langle\psi|\mathbf{E}(\mathbf{r})|\psi\rangle = \frac{1}{3}\left(\langle 1,\mathbf{q},\tau| - i\sqrt{2}\langle 2,\mathbf{q},\tau|\right)\sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\,i\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}} - a^\dagger_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma}e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$
$$\left(|1,\mathbf{q},\tau\rangle + i\sqrt{2}\,|2,\mathbf{q},\tau\rangle\right)$$

The only terms that will contribute are when we are creating or annihilating a photon of momentum $\mathbf{q}$ and polarization τ, so we have

$$\langle\psi|\mathbf{E}(\mathbf{r})|\psi\rangle = \frac{i}{3}\sqrt{\frac{\hbar\omega_q}{2\varepsilon_0 V}}\left(\langle 1,\mathbf{q},\tau| - i\sqrt{2}\langle 2,\mathbf{q},\tau|\right)\left(a_{\mathbf{q}\tau}\boldsymbol{\varepsilon}_{\mathbf{q}\tau}e^{i\mathbf{q}\cdot\mathbf{r}} - a^\dagger_{\mathbf{q}\tau}\boldsymbol{\varepsilon}^*_{\mathbf{q}\tau}e^{-i\mathbf{q}\cdot\mathbf{r}}\right)\left(|1,\mathbf{q},\tau\rangle + i\sqrt{2}\,|2,\mathbf{q},\tau\rangle\right)$$

The matrix elements are non-vanishing only if you increase from 1 to 2 or decrease from 2 to 1. In each case the resulting matrix element is $\sqrt{2}$, so we have

$$\langle\psi|\mathbf{E}(\mathbf{r})|\psi\rangle = \frac{i}{3}\sqrt{\frac{\hbar\omega_q}{2\varepsilon_0 V}}\left[i\sqrt{2}\langle 2,\mathbf{q},\tau|a^\dagger_{\mathbf{q}\tau}\boldsymbol{\varepsilon}^*_{\mathbf{q}\tau}e^{-i\mathbf{q}\cdot\mathbf{r}}|1,\mathbf{q},\tau\rangle + \langle 1,\mathbf{q},\tau|a_{\mathbf{q}\tau}\boldsymbol{\varepsilon}_{\mathbf{q}\tau}e^{i\mathbf{q}\cdot\mathbf{r}}i\sqrt{2}\,|2,\mathbf{q},\tau\rangle\right]$$
$$= -\frac{1}{3}\sqrt{\frac{\hbar\omega_q}{2\varepsilon_0 V}}\left[2\boldsymbol{\varepsilon}^*_{\mathbf{q}\tau}e^{-i\mathbf{q}\cdot\mathbf{r}} + 2\boldsymbol{\varepsilon}_{\mathbf{q}\tau}e^{i\mathbf{q}\cdot\mathbf{r}}\right].$$

We then substitute $\mathbf{q} = q\hat{\mathbf{x}}$, $\boldsymbol{\varepsilon}_\tau = \hat{\mathbf{y}}$, and $\omega_q = cq$ to yield

$$\langle\psi|\mathbf{E}(\mathbf{r})|\psi\rangle = -\frac{2}{3}\sqrt{\frac{\hbar cq}{2\varepsilon_0 V}}\left(\hat{\mathbf{y}}e^{-iqx} + \hat{\mathbf{y}}e^{-iqx}\right) = -\frac{4}{3}\sqrt{\frac{\hbar cq}{2\varepsilon_0 V}}\hat{\mathbf{y}}\cos(qx).$$

5. **An electron of mass *m* is in a 3D symmetric harmonic oscillator with frequency ω in the superposition state $|2,0,0\rangle$. Assume the dipole approximation is valid.**

(a) [5] To which state(s) will it decay? Calculate the relevant matrix elements $\mathbf{r}_{FI}$.

In the dipole approximation, the rate is proportional to the square of $\mathbf{r}_{FI} = \langle \phi_F | \mathbf{R} | \phi_I \rangle$. The three position operators can only raise or lower one of the three labels of the eigenstates. But we need to go down in energy, not up, so only the final state that is relevant will be $|1,0,0\rangle$, for which the only non-zero matrix element is

$$\mathbf{r}_{FI} = \langle 1,0,0|\mathbf{R}|2,0,0\rangle = \hat{\mathbf{x}}\langle 1,0,0|X|2,0,0\rangle = \hat{\mathbf{x}}\sqrt{\frac{\hbar}{2m\omega}}\langle 1,0,0|\left(a+a^\dagger\right)|2,0,0\rangle = \hat{\mathbf{x}}\sqrt{\frac{\hbar}{2m\omega}}\sqrt{2}.$$

(b) [7] Calculate the differential decay rates $d\Gamma_1/d\Omega$ and $d\Gamma_2/d\Omega$ to the two polarization states $\boldsymbol{\varepsilon}_1 = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$ and $\boldsymbol{\varepsilon}_2 = (-\sin\phi, \cos\phi, 0)$.

We need the frequency difference, which is

$$\omega_{FI} = \hbar^{-1}\left[\hbar\omega\left(2+0+0+\tfrac{3}{2}\right) - \hbar\omega\left(1+0+0+\tfrac{3}{2}\right)\right] = \omega.$$

The differential decay rates are given by the formula

$$\frac{d\Gamma_i}{d\Omega} = \frac{\alpha\omega_{IF}^3}{2\pi c^2}\left|\boldsymbol{\varepsilon}_i^* \cdot \mathbf{r}_{FI}\right|^2 = \frac{\alpha\omega^3}{2\pi c^2}\left(\frac{\hbar}{m\omega}\right)\left|\boldsymbol{\varepsilon}_i^* \cdot \hat{\mathbf{x}}\right|^2 = \frac{\alpha\hbar\omega^2}{2\pi mc^2}\left|\varepsilon_{ix}^*\right|^2$$

We just apply it to the two polarizations which are

$$\frac{d\Gamma_1}{d\Omega} = \frac{\alpha\hbar\omega^2}{2\pi mc^2}\cos^2\theta\cos^2\phi, \quad \frac{d\Gamma_2}{d\Omega} = \frac{\alpha\hbar\omega^2}{2\pi mc^2}\sin^2\phi.$$

(c) [8] Integrate over angles and determine the polarized decay rates Γ_1 and Γ_2, and the branching ratio to each of these polarizations.

Integrating over angles, we have

$$\Gamma_1 = \int \frac{d\Gamma_1}{d\Omega}d\Omega = \frac{\alpha\hbar\omega^2}{2\pi mc^2}\int_{-1}^{1}\cos^2\theta\, d\left(\cos\theta\right)\int_0^{2\pi}\cos^2\phi\, d\phi = \frac{\alpha\hbar\omega^2}{2\pi mc^2}\cdot\frac{2}{3}\pi = \frac{\alpha\hbar\omega^2}{3mc^2},$$

$$\Gamma_2 = \int \frac{d\Gamma_2}{d\Omega}d\Omega = \frac{\alpha\hbar\omega^2}{2\pi mc^2}\int_{-1}^{1} d\left(\cos\theta\right)\int_0^{2\pi}\sin^2\phi\, d\phi = \frac{\alpha\hbar\omega^2}{2\pi mc^2}\cdot 2\pi = \frac{\alpha\hbar\omega^2}{mc^2}.$$

The total decay rate is $\Gamma = \Gamma_1 + \Gamma_2 = 4\alpha\hbar\omega^2/3mc^2$, which we also could have easily found from the formula $\Gamma = 4\alpha\omega_{IF}^3\left|\mathbf{r}_{FI}\right|^2/3c^2$. The branching ratios are each part divided by the total, so

$$\mathrm{BR}_1 = \frac{\alpha\hbar\omega^2}{3mc^2}\cdot\frac{3mc^2}{4\alpha\hbar\omega^2} = \frac{1}{4}, \quad \mathrm{BR}_2 = \frac{\alpha\hbar\omega^2}{mc^2}\cdot\frac{3mc^2}{4\alpha\hbar\omega^2} = \frac{3}{4}.$$

1D Harmonic Oscillator: $X = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right),$

$$\phi_0\left(x\right) = \left(\frac{\alpha}{\pi}\right)^{1/4} \exp\left(\frac{-\alpha x^2}{2}\right), \quad \phi_1\left(x\right) = \left(\frac{\alpha}{\pi}\right)^{1/4} \sqrt{2\alpha}\, x \exp\left(\frac{-\alpha x^2}{2}\right), \quad \alpha = \frac{m\omega}{\hbar}.$$

First Order Time-dependent Perturbation Theory: $S_{FI} = \delta_{FI} + \frac{1}{i\hbar}\int^{T} W_{FI}\left(t\right) e^{i\omega_{FI} t} dt + \cdots$

Electric Field: $\mathbf{E}\left(\mathbf{r}\right) = \sum_{\mathbf{k},\sigma} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\, i\left(a_{\mathbf{k}\sigma} \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r}} - a^\dagger_{\mathbf{k}\sigma} \boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma} e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$

Spontaneous Decay: $\frac{d\Gamma}{d\Omega} = \frac{\alpha\omega_{IF}^3}{2\pi c^2}\left|\boldsymbol{\varepsilon}^* \cdot \mathbf{r}_{FI}\right|^2, \quad \Gamma = \frac{4\alpha\omega_{IF}^3}{3c^2}\left|\mathbf{r}_{FI}\right|^2$

Possibly Helpful Integrals:

$$\int_0^\infty x^n e^{-Ax^2} dx = \tfrac{1}{2} A^{-(n+1)/2} \Gamma\left(\tfrac{n+1}{2}\right), \quad \Gamma\left(1\right) = \Gamma\left(2\right) = 1, \quad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\sqrt{\pi}, \quad \Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{4}\sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-Ax^2 - Bx} dx = \sqrt{\pi/A}\, e^{B^2/4A}.$$

$$\int_0^{2\pi} \sin^2\phi\, d\phi = \int_0^{2\pi} \cos^2\phi\, d\phi = \pi, \quad \int_0^{\pi} \sin^2\theta\, d\theta = \frac{\pi}{2}, \quad \int_0^{\pi} \sin^3\theta\, d\theta = \frac{4}{3}, \quad \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta = \frac{2}{3}.$$