Physics 741 – Graduate Quantum Mechanics 1

Midterm Exam, Fall 2018

Please note that some possibly helpful formulas and integrals appear on the second page. Note also that there is one problem on the second page. Each question is worth 20 points, with points for each part marked separately.

1. Consider the wave function $\psi(x) = \begin{cases} \sqrt{12\lambda}\left(e^{-\lambda x} - e^{-2\lambda x}\right) & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases}$
   (a) [16] What are $\langle X \rangle$, $\langle P \rangle$, $\langle X^2 \rangle$, and $\langle P^2 \rangle$ for this state?
   (b) [4] Find the uncertainties Δx and Δp, and show they satisfy the uncertainty relation.

2. A particle of mass *m* lies in the infinite square well with allowed region $0 < x < a$. The wave function at $t = 0$ in this region is $\Psi(x,0) = \psi(x) = \left(2/\sqrt{3a}\right)\sin(5\pi x/a)\left[1 + i\cos(2\pi x/a)\right]$.
   (a) [7] Write this state in the form $|\psi\rangle = \sum_n c_n |\phi_n\rangle$, where $|\phi_n\rangle$ are the energy eigenstates. Some helpful formulas are provided on the next page.
   (b) [7] Check the normalization in both the coordinate and eigenstate basis.
   (c) [6] Write the wave function $\Psi(x,t)$ at all times.

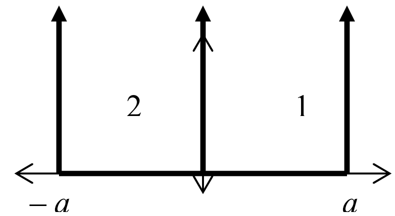
3. A particle of mass *m* lies in the potential $V(x,y,z) = \alpha\left(x^2 + y^2 + z^2\right)^2 + \gamma\left(x^2 y + y^2 z + z^2 x\right)$. Consider the rotation operator that rotates the three coordinates among each other, so that $\mathcal{R}(x,y,z) = (y,z,x)$, *i.e.* $x' = y, y' = z, z' = x$.
   (a) [6] Show that this is a symmetry operation; that is, *V* is unchanged by this transformation. You may assume that the kinetic term in the Hamiltonian is also unchanged.
   (b) [7] Argue that if this symmetry operation were performed a particular number of times, the resulting symmetry operation would correspond with the identity operation. How many times?
   (c) [7] Argue that eigenstates of the Hamiltonian can be chosen to also be eigenstates of this symmetry operation. What are the possible eigenvalues of these states under the symmetry operation?

4. A particle is in the state $|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, when the operator $B = b\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is measured.

   What are the possible outcomes, and what would be the corresponding probabilities? For each of these possible outcomes, what would be the state vector after measurrement?

5. A particle of mass m is in an infinite square well with a spike in the middle, with potential

$$V(x)=\begin{cases}\infty & \text{if } |x|>a\,,\\ \lambda\delta(x) & \text{if } |x|<a\,.\end{cases}$$



This potential is sketched at right. In region 1, the solution to Schrödinger's equation is $\psi_1=\sin(ka-kx)$ with energy $E=\hbar^2k^2/2ma^2$.

(a) [7] Argue based on symmetry that there are "even" and "odd" solutions. For the odd solutions, argue that $\psi(0)=0$. Find all possible values for *k* in this case. For the even solutions, show in region 2 that $\psi_2(x)=\sin(ka+kx)$.

(b) [7] Integrate Schrödinger's equation from $-\varepsilon$ to $+\varepsilon$ across the origin, in the limit $\varepsilon\to 0^+$ to find a formula for the discontinuity in the derivative.

(c) [6] For the even solutions, use this to find a formula for $k\cot(ka)$.

Possibly Helpful Formulas:

<u>Infinite Square Well</u>
mass *m*, region $0<x<a$

$$\psi_n(x)=\sqrt{2/a}\,\sin(\pi nx/a)$$

$$E_n=\frac{n^2\pi^2\hbar^2}{2ma^2}$$

Possibly Helpful Integrals: *n*, *p* and *q* are assumed to be positive integers

$$\int_0^\infty x^n e^{-\alpha x}dx=\frac{n!}{\alpha^{n+1}},\qquad \int_0^a \sin\left(\frac{\pi nx}{a}\right)dx=\begin{cases}\frac{2a}{\pi n} & \text{if } n \text{ odd,}\\ 0 & \text{if } n \text{ even.}\end{cases}\qquad \int_0^a \sin\left(\frac{\pi nx}{a}\right)\sin\left(\frac{\pi px}{a}\right)dx=\tfrac{1}{2}a\delta_{np}\,,$$

$$\int_0^a \sin\left(\frac{\pi nx}{a}\right)\cos\left(\frac{\pi px}{a}\right)dx=\begin{cases}\frac{2an}{\pi(n^2-p^2)} & \text{if } n+p \text{ odd,}\\ 0 & \text{if } n+p \text{ even.}\end{cases}$$

$$\int_0^a \sin\left(\frac{\pi nx}{a}\right)\sin\left(\frac{\pi px}{a}\right)\cos\left(\frac{\pi qx}{a}\right)dx=\tfrac{1}{4}a\left(\delta_{n,p+q}+\delta_{p,n+q}-\delta_{q,n+p}\right),$$

$$\int_0^a \sin^2\left(\frac{\pi nx}{a}\right)\cos^2\left(\frac{\pi px}{a}\right)dx=a\left(\tfrac{1}{4}-\tfrac{1}{8}\delta_{np}\right)\qquad \int_0^a \cos\left(\frac{\pi nx}{a}\right)\sin^2\left(\frac{\pi px}{a}\right)dx=\tfrac{1}{4}a\delta_{n,2p}\,.$$