Physics 741 – Graduate Quantum Mechanics 1

Solutions to Final Exam, Fall 2016

Each question is worth 25 points, with points for each part marked separately. Some possibly useful formulas can be found at the end of the exam.

1. The Σ^{*0} baryon is a particle made of three quarks, one each up (u), down (d), and strange (s). These quarks are each spin $\frac{1}{2}$, and they are believed to have no orbital angular momentum.

(a) [10] Based only on the spins of the quarks, if we measured $\mathbf{S}_u^2$, the spin on just the up quark what would we get? What could we possibly get if we measured the sum of the up and down quarks, $\mathbf{S}_{ud}^2$, where $\mathbf{S}_{ud} = \mathbf{S}_u + \mathbf{S}_d$? What if we measured the total spin squared $\mathbf{S}^2$, where $\mathbf{S} = \mathbf{S}_u + \mathbf{S}_d + \mathbf{S}_s$?

The eigenvalue of any angular momentum operator is $\mathbf{J}^2 = \hbar^2\left(j^2 + j\right)$. For just the up quark, since $s = \frac{1}{2}$, we have $\mathbf{S}_u^2 = \hbar^2\left[\left(\frac{1}{2}\right)^2 + \frac{1}{2}\right] = \frac{3}{4}\hbar^2$. When we add two spins, we can get total spin from $\left|\frac{1}{2} - \frac{1}{2}\right|$ to $\frac{1}{2} + \frac{1}{2}$, so it is 0 or 1. Hence we would have $\mathbf{S}_{ud}^2 = 0$ or $\mathbf{S}_{ud}^2 = 2\hbar^2$. Finally, when we add the strange quark, we would be adding spin 0 or 1 to spin $\frac{1}{2}$. If we are adding 0, we would get total spin running from $\left|0 - \frac{1}{2}\right|$ to $0 + \frac{1}{2}$, so it must be $s = \frac{1}{2}$. If we had spin 1, then it runs from $\left|1 - \frac{1}{2}\right|$ to $1 + \frac{1}{2}$, so the possibilities are $s = \frac{1}{2}, \frac{3}{2}$. Hence the total spin squared will be $\mathbf{S}^2 = \frac{3}{4}\hbar^2, \frac{3}{4}\hbar^2$ or $\frac{15}{4}\hbar^2$. The repetition of the $\frac{3}{4}\hbar^2$ simply signifies that there are two separate ways to make this total.

(b) [8] In fact, the Σ^{*0} is known to be spin $\frac{3}{2}$. In light of this, how, at all, would your answers change to part (a)?

Obviously, the answer $\mathbf{S}_u^2 = \frac{3}{4}\hbar^2$ is unchanged, and the answer for the total is now $\mathbf{S}^2 = \frac{15}{4}\hbar^2$. To get total spin $\frac{3}{2}$ however, at the intermediate step, we *must* have had spin 1, so this implies $\mathbf{S}_{ud}^2 = 2\hbar^2$. They are now all definite.

(c) [7] A particular Σ^{*0} has its spin around the *z*-axis measured, and it is found to have eigenvalue $+\frac{1}{2}\hbar$. If this same Σ^{*0} were to have the *z*-component of the strange quark measured S_{sz} or the combined *z*-component spin of the up and down quarks $S_{ud,z}$, what would be the possible outcomes?

The measurement of S_{sz} for a spin $\frac{1}{2}$ particle is $\pm\frac{1}{2}\hbar$. Since the total *z*-component of angular momentum is $S_z = S_{ud,z} + S_{sz}$, we have $S_{ud,z} = S_z - S_{sz}$, and hence the eigenvalue will be $S_{ud,z} = \frac{1}{2}\hbar \mp \frac{1}{2}\hbar$, so that is must be either 0 or $\hbar$.

2. **An electron is in a region with scalar potential $U = -\frac{1}{2}c\left(x^2+y^2\right)$ and vector potential $\mathbf{A} = \frac{1}{2}b\left(x^2+y^2\right)\hat{\mathbf{z}}$**

(a) **[5] Find the electric and magnetic field from these potentials**

We use the formula

$$\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A} - \nabla U = 0 - \hat{\mathbf{x}}\frac{\partial U}{\partial x} - \hat{\mathbf{y}}\frac{\partial U}{\partial y} - \hat{\mathbf{z}}\frac{\partial U}{\partial z} = cx\hat{\mathbf{x}} + cy\hat{\mathbf{y}},$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}B_z\right) - \hat{\mathbf{y}}\left(\frac{\partial}{\partial x}B_z\right) = \hat{\mathbf{x}}(by) - \hat{\mathbf{y}}(bx) = b\left(y\hat{\mathbf{x}} - x\hat{\mathbf{y}}\right).$$

(b) **[8] Write the Hamiltonian explicitly. Show that it commutes with one of the momentum operators. Which one?**

We first write $\boldsymbol{\pi} = \mathbf{P} + e\mathbf{A}$ and then substitute into the provided formula. We have

$$\begin{aligned}
H &= \frac{\boldsymbol{\pi}^2}{2m} - eU + \frac{ge}{2m}\mathbf{B}\cdot\mathbf{S} = \frac{1}{2m}\left(\mathbf{P}+e\mathbf{A}\right)^2 - eU + \frac{ge}{2m}\mathbf{B}\cdot\mathbf{S} \\
&= \frac{1}{2m}\left\{P_x^2 + P_y^2 + \left[P_z + \tfrac{1}{2}eb\left(X^2+Y^2\right)\right]^2\right\} + \tfrac{1}{2}ec\left(X^2+Y^2\right) + \frac{geb}{2m}\left(YS_x - XS_y\right) \\
&= \frac{1}{2m}\left[\mathbf{P}^2 + ebP_z\left(X^2+Y^2\right) + \tfrac{1}{4}e^2b^2\left(X^2+Y^2\right)^2\right] + \tfrac{1}{2}ec\left(X^2+Y^2\right) + \frac{geb}{2m}\left(YS_x - XS_y\right).
\end{aligned}$$

We note that the operator P_z commutes with every operator appearing in here, so $\left[H, P_z\right] = 0$.

(c) **[12] Find the commutator of L_z and S_z with the Hamiltonian H. Show that only one of L_z, S_z, and $J_z = L_z + S_z$ commutes with H. To save time, you may use the fact that L_z commutes with both $\mathbf{P}^2$ and with X^2+Y^2.**

If you use the additional information that $\left[L_z, P_z\right] = 0$, then it is obvious that everything commutes except possibly the last spin term. For this term, we have

$$\begin{aligned}
\left[L_z, H\right] &= \frac{geb}{2m}\left[L_z, YS_x - XS_y\right] = \frac{geb}{2m}\left(\left[L_z, Y\right]S_x - \left[L_z, X\right]S_y\right) = \frac{geb}{2m}\left(-i\hbar XS_x - i\hbar YS_y\right), \\
\left[S_z, H\right] &= \frac{geb}{2m}\left[S_z, YS_x - XS_y\right] = \frac{geb}{2m}\left(Y\left[S_z, S_x\right] - X\left[S_z, S_y\right]\right) = \frac{geb}{2m}\left(i\hbar YS_y + i\hbar XS_x\right), \\
\left[J_z, H\right] &= \left[L_z, H\right] + \left[S_z, H\right] = 0.
\end{aligned}$$

3. The one dimensional infinite square well with width *a* for spinless particles of mass *m* has eigenstates $|\psi_n\rangle$, which have energy $E_n = \pi^2 n^2 \hbar^2 / (2ma^2)$.

(a) [3] What are the corresponding energies for a three-dimensional infinite square well of side *a*?

The energies are just the sums of the energies in the three directions, so

$$E_{npq} = \frac{\pi^2 \hbar^2}{2ma^2}\left(n^2 + p^2 + q^2\right).$$

(b) [8] Suppose we have 14 spin-0 identical non-interacting particles in this same 3D square well. Would these particles be bosons or fermions? In the ground state, which states would be occupied, and what would be the energy of the ground state?

Spin 0 particles are bosons. The ground state would just involve all the particles being in the same state, so they would all be in the state $|\psi_{111}\rangle$, and the total energy would be 14 times the energy of a single particle, so

$$E_{tot} = 14 \cdot \frac{3\pi^2 \hbar^2}{2ma^2} = \frac{21\pi^2 \hbar^2}{ma^2}.$$

(c) [14] Suppose we have 14 spin-½ identical non-interacting particles in this same 3D square well. Would these particles be bosons or fermions? In the ground state, which states would be occupied, and what would be the energy of the ground state?

Because these states have spin, a single particle would now be denoted by $|\psi_{npq}, \chi\rangle$, where $\chi = \pm$ is the spin state. Because of the Pauli exclusion principle, you can only put one particle in each state, but we can put them in different spin states, so we will fill in the lowest seven energy states with both spins. So the occupied states are

$$\left\{|\psi_{111},\pm\rangle, |\psi_{112},\pm\rangle, |\psi_{121},\pm\rangle, |\psi_{211},\pm\rangle, |\psi_{122},\pm\rangle, |\psi_{212},\pm\rangle, |\psi_{221},\pm\rangle\right\}$$

The first state has $n^2 + p^2 + q^2 = 3$, the next three have $n^2 + p^2 + q^2 = 6$, and the last three have $n^2 + p^2 + q^2 = 9$, so in total the energy, remembering to double for each spin state, is

$$E_{tot} = \frac{\pi^2 \hbar^2}{2ma^2}\left(1\cdot 3 + 3\cdot 6 + 3\cdot 9\right)\cdot 2 = \frac{48\pi^2 \hbar^2}{ma^2}.$$

4. Two electrons are in one dimension, one each in the two orthonormal wave functions $\phi_1(x)=\sqrt{\lambda}e^{-\lambda|x|}$ and $\phi_2(x)=\sqrt{2\lambda^3}\,xe^{-\lambda|x|}$. However, these wave functions do not take into account the electron spin, nor do they take into account the fact that they are fermions.

(a) [9] Assume the two particles are in one of the two spin states $|\chi_\pm\rangle=\frac{1}{\sqrt{2}}\left(|+-\rangle\pm|-+\rangle\right)$. In each case, write out explicitly the properly normalized space wave function $\psi(x_1,x_2)$.

It is easily seen that the spin states have been properly normalized. Note that the spin part is either symmetric or anti-symmetric, depending on the sign of the $\pm$. The space part must be the opposite to compensate, so for the spin state $|\chi_\pm\rangle$, the space part must be $|\psi_+\rangle=\frac{1}{\sqrt{2}}\left(|\phi_1\phi_2\rangle\mp|\phi_2\phi_1\rangle\right)$. Note that these are properly normalized. Hence the wave function must be

$$\psi_\pm(x_1,x_2)=\langle x_1,x_2|\psi_\pm\rangle=\tfrac{1}{\sqrt{2}}\left[\phi_1(x_1)\phi_2(x_2)\mp\phi_1(x_2)\phi_2(x_1)\right]=\lambda^2(x_1\mp x_2)e^{-\lambda|x_1|}e^{-\lambda|x_2|}.$$

(b) [8] For each case, find the probability that the first particle is at a positive position, $x_1>0$.

We integrate the probability density over two dimensions for the whole region $x_1>0$, so

$$\begin{aligned}P(x_1>0)&=\int_0^\infty dx_1\int_{-\infty}^\infty dx_2\,|\psi(x_1,x_2)|^2=\lambda^4\int_0^\infty e^{-2\lambda|x_1|}dx_1\int_{-\infty}^\infty(x_1\mp x_2)^2e^{-2\lambda|x_2|}dx_2\\&=\lambda^4\int_0^\infty e^{-2\lambda x_1}dx_1\int_{-\infty}^\infty\left(x_1^2\mp 2x_1x_2+x_2^2\right)^2e^{-2\lambda|x_2|}dx_2\\&=2\lambda^4\left\{\int_0^\infty x_1^2e^{-2\lambda x_1}dx_1\int_{-\infty}^\infty e^{-2\lambda x_2}dx_2\mp 0+\int_0^\infty e^{-2\lambda x_1}dx_1\int_{-\infty}^\infty x_2^2e^{-2\lambda x_2}dx_2\right\}\\&=2\lambda^4\left\{2(2\lambda)^{-3}(2\lambda)^{-1}+(2\lambda)^{-1}2(2\lambda)^{-3}\right\}=2\left\{\tfrac{2}{16}+\tfrac{2}{16}\right\}=\tfrac{8}{16}=\tfrac{1}{2}.\end{aligned}$$

The cross term vanished because the x_2 integral is an integral of an odd function over an even interval.

(c) [8] For each case, find the probability that both particles are at a positive position, $x_1,x_2>0$.

The calculation is the same as before, but with a different region of integration:

$$\begin{aligned}P(x_1,x_2>0)&=\int_0^\infty dx_1\int_0^\infty dx_2\,|\psi(x_1,x_2)|^2=\lambda^4\int_0^\infty e^{-2\lambda|x_1|}dx_1\int_0^\infty(x_1\mp x_2)^2e^{-2\lambda|x_2|}dx_2\\&=\lambda^4\int_0^\infty e^{-2\lambda x_1}dx_1\int_0^\infty\left(x_1^2\mp 2x_1x_2+x_2^2\right)e^{-2\lambda x_2}dx_2\\&=\lambda^4\left\{2(2\lambda)^{-3}(2\lambda)^{-1}\mp 2(2\lambda)^{-2}(2\lambda)^{-2}+(2\lambda)^{-1}2(2\lambda)^{-3}\right\}=\tfrac{2}{16}\mp\tfrac{2}{16}+\tfrac{2}{16}=\tfrac{2\mp 1}{8}.\end{aligned}$$

This works out to $\frac{1}{8} = 12.5\%$ for $|\chi_+\rangle$ and $\frac{3}{8} = 37.5\%$ for $|\chi_-\rangle$.

5. A particle is in the state $|\psi\rangle = \begin{pmatrix} \cos\left(\frac{1}{2}\theta\right) \\ \sin\left(\frac{1}{2}\theta\right) \end{pmatrix}$, but the angle θ is unknown, and has a $\frac{1}{3}$ chance of being in each of the values $\theta = 0, \frac{1}{2}\pi, \pi$.

(a) [8] Find the state operator ρ.

We simply use the formula

$$\rho = \sum_i f_i |\psi_i\rangle\langle\psi_i| = \frac{1}{3}\left[\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$

$$= \frac{1}{3}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \frac{1}{3}\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{2} \end{pmatrix}.$$

(b) [9] Find the expectation value of the operators S_x and S_z for this state operator.

To find these, we simply take the trace of the product of S_i with ρ, so we have

$$\langle S_x \rangle = \mathrm{Tr}(\rho S_x) = \tfrac{1}{2}\hbar\,\mathrm{Tr}\left[\begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \tfrac{1}{2}\hbar\,\mathrm{Tr}\begin{pmatrix} \frac{1}{6} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} \end{pmatrix} = \tfrac{1}{2}\hbar\tfrac{1}{3} = \tfrac{1}{6}\hbar,$$

$$\langle S_z \rangle = \mathrm{Tr}(\rho S_z) = \tfrac{1}{2}\hbar\,\mathrm{Tr}\left[\begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \tfrac{1}{2}\hbar\,\mathrm{Tr}\begin{pmatrix} \frac{1}{2} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{1}{2} \end{pmatrix} = \tfrac{1}{2}\hbar 0 = 0.$$

(c) [8] Show that if the Hamiltonian is $H = aS_x$, the state operator will be time independent.

The time derivative of the state operator is given by

$$i\hbar\frac{d\rho}{dt} = [H, \rho] = \tfrac{1}{2}\hbar a[\sigma_x, \rho] = \tfrac{1}{2}\hbar a\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$$

$$= \tfrac{1}{2}\hbar a\left\{\begin{pmatrix} \frac{1}{6} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} \end{pmatrix} - \begin{pmatrix} \frac{1}{6} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} \end{pmatrix}\right\} = 0.$$

Since the time derivative vanishes, it will not change (and it will continue to not change).

6. **This problem is to be worked entirely in the Hamiltonian formulation of quantum mechanics. Consider a particle of mass *m* in the one-dimensional linear potential $V(X) = -aX$**

   (a) **[9] What is the Hamiltonian? Find expressions for the derivatives of the position operator *X* and momentum operator *P*.**

The Hamiltonian is, of course, just

$$H = \frac{1}{2m}P^2 + V(X) = \frac{1}{2m}P^2 - aX.$$

Keep in mind that all of these operators are time-dependent in this formalism. The operators evolve according to

$$\frac{d}{dt}X = \frac{i}{\hbar}[H, X] = \frac{i}{2\hbar m}\left[P^2, X\right] = \frac{i}{2\hbar m}\left(P[P, X] + [P, X]P\right) = \frac{i}{2\hbar m}\left(-i\hbar P - i\hbar P\right) = \frac{1}{m}P,$$

$$\frac{d}{dt}P = \frac{i}{\hbar}[H, P] = -\frac{ia}{\hbar}[X, P] = \frac{-ii\hbar a}{\hbar} = a.$$

   (b) **[9] Solve for the position and momentum operators at time *t* in terms of the operators at time 0.**

The second one is easy to solve. We simply integrate the trivial equation and match the boundary condition at $t = 0$, so we have

$$P(t) = at + P(0).$$

We then substitute this into the other equation, and integrate again, again using the boundary conditions to match at $X(0)$:

$$\frac{d}{dt}X = \frac{1}{m}P = \frac{1}{m}\left[P(0) + at\right],$$

$$X(t) = \frac{at^2}{2m} + \frac{1}{m}P(0)t + X(0).$$

   (c) **[7] Show that there is a minimum uncertainty relation between the uncertainty of the initial position $\Delta x(0)$ and the position at time *t*, $\Delta x(t)$.**

We now use the generalized uncertain principle to get a relationship between the uncertainty of $X(0)$ and $X(t)$. Note that the first term in $X(t)$ has no operators, so it commutes with everything.

$$\left[\Delta x(0)\right]\left[\Delta x(t)\right] \geq \tfrac{1}{2}\left|\left\langle i\left[X(0), X(t)\right]\right\rangle\right| = \tfrac{1}{2}\left|\left\langle i\left[X(0), \frac{at^2}{2m} + \frac{1}{m}P(0)t + X(0)\right]\right\rangle\right| \frac{d}{dt}X$$

$$= \frac{1}{2m}\left|\left\langle it\left[X(0), P(0)\right]\right\rangle\right| = \frac{1}{2m}\left|i^2\hbar t\right| = \frac{\hbar|t|}{2m}.$$

Possibly Useful Formulas

Spin-½: $\mathbf{S} = \frac{1}{2}\hbar\boldsymbol{\sigma}$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

EM Fields

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\partial \mathbf{A}/\partial t - \nabla U$$

State Operator

$$\rho = \sum_i f_i |\psi_i\rangle\langle\psi_i|$$

$$i\hbar \frac{d\rho}{dt} = [H, \rho]$$

Generalized Uncertainty

$$(\Delta a)(\Delta b) \geq \tfrac{1}{2}\left|\langle i[A,B]\rangle\right|$$

Heisenberg Evolution

$$\frac{d}{dt} A = \frac{i}{\hbar}[H, A]$$

EM Hamiltonian

$$H = \frac{\boldsymbol{\pi}^2}{2m} - eU + \frac{ge}{2m}\mathbf{B}\cdot\mathbf{S}$$

Possibly Useful Integral: $\int_0^\infty x^n e^{-\alpha x} dx = \alpha^{-n-1} n!$