Physics 741 – Graduate Quantum Mechanics 1

Midterm Exam, Fall 2016

Please note that some possibly helpful formulas and integrals appear on the second page. Note also that there is one problem on the second page. Each question is worth 20 points, with points for each part marked separately.

1. Consider the wave function $\psi(x) = \begin{cases} Nx(a-x) & 0 < x < a\,, \\ 0 & \text{otherwise.} \end{cases}$

   Once properly normalized, this wave function has $\langle X \rangle = \frac{1}{2}a$ and $\langle X^2 \rangle = \frac{2}{7}a^2$.

   (a) [5] What is the correct normalization N?

   (b) [8] What are $\langle P \rangle$ and $\langle P^2 \rangle$ for this state?

   (c) [7] Find the uncertainties Δx and Δp and show that they satisfy the uncertainty relation.

2. A particle of mass m lies in the infinite square well with allowed region $0 < x < a$. The wave function takes the form $\psi(x) = \begin{cases} N\sin^2(\pi x/a) & 0 < x < a\,, \\ 0 & \text{elsewhere.} \end{cases}$

   (a) [5] Determine the normalization constant N.

   (b) [7] Write this state in the form $|\psi\rangle = \sum_n c_n |\phi_n\rangle$, where $|\phi_n\rangle$ are the energy eigenstates. Some helpful integrals are provided.

   (c) [8] If we were to measure the energy, what would be the possible outcomes and corresponding probabilities? Give a general formula, and find the numeric value as a percentage for the first three non-zero outcomes.

3. Consider the harmonic oscillator with mass m and angular frequency ω. At $t = 0$, the system is in the state $|\Psi(t=0)\rangle = N\sum_{n=1}^{\infty} \frac{i^n}{n^2} |n\rangle$.

   (a) [7] What is the correct normalization N? Some helpful sums are given on the next page.

   (b) [5] Find the value of $\langle P \rangle$ for this state. Simplify as much as possible.

   (c) [8] What is $|\Psi(t)\rangle$ at all times?

4. A hydrogen atom is in the state $|n,l,m\rangle = |2,1,0\rangle$.

   (a) [6] What would be the result if you measure the energy, orbital angular momentum squared L^2 and z-component L_z?

   (b) [6] Write the explicit form of the wave function $\psi(r,\theta,\phi)$.

   (c) [8] Calculate the expectation value $\langle R^{-1} \rangle$ for this wave function, where R is the distance from the origin operator.

5. In a certain basis, the state vector is given by $|\Psi\rangle = \begin{pmatrix} \frac{1}{3}+\frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$, and the spin operator in the x-directions is given by $S_x = \frac{1}{2}\hbar\sigma_x = \frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(a) [10] Find the eigenvalues and normalized eigenvectors of S_x.

(b) [10] If you measured S_x, what is the probability that you get each of the possible eigenvalues you found in part (a)? What would be the state vector afterwards?

Possibly Helpful Formulas:

Harmonic Oscillator	Radial Wave Functions	Spherical Harmonics	Hydrogen Energy
$X = \sqrt{\frac{\hbar}{2m\omega}}\left(a+a^\dagger\right)$	$R_{10}(r) = \frac{2e^{-r/a_0}}{\sqrt{a^3}}$	$Y_1^0(\theta,\phi) = \frac{\sqrt{3}}{2\sqrt{\pi}}\cos\theta$	$E = -\frac{13.6\text{ eV}}{n^2}$
$P = i\sqrt{\frac{1}{2}\hbar m\omega}\left(a^\dagger - a\right)$	$R_{20}(r) = \frac{e^{-r/2a}}{\sqrt{2a^3}}\left(1-\frac{r}{2a}\right)$	$Y_2^0(\theta,\phi) = \frac{\sqrt{5}}{4\sqrt{\pi}}\left(3\cos^2\theta - 1\right)$	
$a\lvert n\rangle = \sqrt{n}\lvert n-1\rangle$			
$a^\dagger\lvert n\rangle = \sqrt{n+1}\lvert n+1\rangle$	$R_{21}(r) = \frac{re^{-r/2a}}{2\sqrt{6a^5}}$	$Y_2^{\pm1}(\theta,\phi) = \mp\frac{\sqrt{15}\,e^{\pm i\phi}}{2\sqrt{2\pi}}\sin\theta\cos\theta$	

Possibly Helpful Integrals: n and p are assumed to be positive integers

$$\int_0^\infty x^n e^{-\alpha x}dx = \frac{n!}{\alpha^{n+1}}, \qquad \int_0^a \sin\left(\frac{\pi n x}{a}\right)dx = \begin{cases} \frac{2a}{\pi n} & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even.} \end{cases} \qquad \int_0^a \sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi p x}{a}\right)dx = \tfrac{1}{2}a\delta_{np}\,.$$

$$\int_0^a \sin\left(\frac{\pi n x}{a}\right)\sin^2\left(\frac{\pi p x}{a}\right)dx = \begin{cases} \frac{4p^2 a}{\pi n\left(4p^2-n^2\right)} & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even.} \end{cases} \qquad \int_0^a \sin^2\left(\frac{\pi n x}{a}\right)\sin^2\left(\frac{\pi p x}{a}\right)dx = a\left(\tfrac{1}{4}+\tfrac{1}{8}\delta_{np}\right).$$

Possibly helpful sums:

$$\sum_{n=1}^\infty \frac{1}{n^k} = \zeta(k), \quad \sum_{n=1}^\infty \frac{(-1)^k}{n^k} = \left(2^{1-k}-1\right)\zeta(k), \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}.$$