

Physics 741 – Graduate Quantum Mechanics 1
Solutions to Final Exam, Fall 2015

Each question is worth 25 points, with points for each part marked separately. Some possibly useful formulas appear at the end of the test.

1. Charmonium is a bound state of a charm quark (spin $s_1 = \frac{1}{2}$) and anti-charm quark (spin $s_2 = \frac{1}{2}$). Although there are many bound states of charmonium, the lowest energy states have orbital angular momentum $l = 0$ or $l = 1$.

(a) [3] What are the possible values for the total spin s of charmonium, corresponding to the total spin $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$?

The total spin runs in integer steps from $s = |s_1 - s_2| = 0$ to $s = s_1 + s_2 = 1$, so in this case, the total spin is just $s = 0$ or 1 .

(b)[8] For each of the cases in part (a), and for each of the cases $l = 0$ or $l = 1$, what is the possible value of j of charmonium, corresponding to the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2$?

In a similar way, the total angular momentum j runs from $j = |l - s|$ to $j = l + s$. There are four cases. In each case, the possible values of j are given in the table at right.

l	s	j
0	0	0
0	1	1
1	0	1
1	1	0,1,2

(c) [10] Suppose one of the states found in part (b) is found to be an eigenstate of J_z with eigenvalue $+2\hbar$. Predict what possible eigenvalues the following operators might have: $\mathbf{J}^2, \mathbf{L}^2, L_z, \mathbf{S}^2, S_z, S_{1z}, S_{2z}$.

The state has $m_j = +2$, and since $|m_j| \leq j$, it follows that $j \geq 2$. But since this is the largest possible value of j , we must have $j = 2$, which in turn implies $l = 1$ and $s = 1$.

Now, since l has a maximum value of 1, and $|m_l| \leq l$, it follows that m_l is at most +1. Similarly, each of the spins has eigenvalues $m_{s_i} = \pm \frac{1}{2}$, so they have maximum values $m_{s_i} = \frac{1}{2}$. Since $J_z = L_z + S_{1z} + S_{2z}$, we have $m_j = m_l + m_{s_1} + m_{s_2}$, and this has a maximum eigenvalue of $m_j = 1 + \frac{1}{2} + \frac{1}{2} = 2$. This corresponds to a J_z eigenvalue of $+2\hbar$, and since we are achieving the maximum, they must all have the maximum value. Finally, all angular momentum squared operators have eigenvalues like $\mathbf{J}^2 = \hbar^2 (j^2 + j)$ while we also have $J_z = \hbar m_j$, so in summary:

$$\mathbf{J}^2 = 6\hbar^2, \quad \mathbf{L}^2 = 2\hbar^2, \quad L_z = +\hbar, \quad \mathbf{S}^2 = 2\hbar^2, \quad S_z = +\hbar, \quad S_{1z} = S_{2z} = \frac{1}{2}\hbar.$$

- (d) [4] Is the charm quark a fermion or boson? What about the anti-charm quark? What about the composite charmonium particles?

Because the charm quark and anti-charm quark have spin $\frac{1}{2}$, they are both fermions. Charmonium, having a total angular momentum that is an integer, is a boson.

2. A sodium atom in the $3P_{3/2}$ state decays by emitting a photon to end up in the $2S_{1/2}$ state. The rate for the decay to a particular polarization can be written in terms of the matrix elements

$$\langle 3S, \frac{1}{2}, m | R_q^{(1)} | 3P, \frac{3}{2}, m' \rangle$$

where $\frac{1}{2}$ and $\frac{3}{2}$ denote the total angular quantum number j for the initial and final atom, m and m' correspond to the z -component of the initial and final angular momenta, and $R_q^{(1)}$ are a set of three rank-one spherical tensors, with $q \in \{-1, 0, 1\}$.

- (a) [13] Give a complete list of the values of the triplets (m, q, m') of this form that do not vanish.

The Wigner-Eckart theorem tells us that

$$\langle 3S, \frac{1}{2}, m | R_q^{(1)} | 3P, \frac{3}{2}, m' \rangle = \frac{1}{\sqrt{2 \cdot \frac{1}{2} + 1}} \langle 3S, \frac{1}{2} || R^{(1)} || 3P, \frac{3}{2} \rangle \langle \frac{3}{2}, 1; m', q | \frac{1}{2}, m \rangle.$$

The Clebsch-Gordan coefficients are nonzero only if $|m| \leq \frac{1}{2}$, $|q| \leq 1$, $|m'| \leq \frac{3}{2}$, and $m = m' + q$. If we pick q arbitrarily from the three choices, $q = -1, 0, 1$ and m from $m = \pm \frac{1}{2}$, then we pick m' so that $m' = m - q$, and we will automatically get m' in the correct range. The table at right lists all appropriate possibilities.

m	q	m'
$+\frac{1}{2}$	$+1$	$-\frac{1}{2}$
$+\frac{1}{2}$	0	$+\frac{1}{2}$
$+\frac{1}{2}$	-1	$+\frac{3}{2}$
$-\frac{1}{2}$	$+1$	$-\frac{3}{2}$
$-\frac{1}{2}$	0	$-\frac{1}{2}$
$-\frac{1}{2}$	-1	$+\frac{1}{2}$

- (b) [12] Suppose the reduced matrix element were known, so that $\langle 3S, \frac{1}{2} || R^{(1)} || 3P, \frac{3}{2} \rangle = A$.

Write the matrix elements from part (a) in terms of A and appropriate Clebsch-Gordan coefficients.

$$\begin{aligned} \langle 3S, \frac{1}{2}, +\frac{1}{2} | R_{+1}^{(1)} | 3P, \frac{3}{2}, -\frac{1}{2} \rangle &= \frac{1}{\sqrt{2}} A \langle \frac{3}{2}, 1; -\frac{1}{2}, +1 | \frac{1}{2}, +\frac{1}{2} \rangle, \\ \langle 3S, \frac{1}{2}, +\frac{1}{2} | R_0^{(1)} | 3P, \frac{3}{2}, +\frac{1}{2} \rangle &= \frac{1}{\sqrt{2}} A \langle \frac{3}{2}, 1; +\frac{1}{2}, 0 | \frac{1}{2}, +\frac{1}{2} \rangle, \\ \langle 3S, \frac{1}{2}, +\frac{1}{2} | R_{-1}^{(1)} | 3P, \frac{3}{2}, +\frac{3}{2} \rangle &= \frac{1}{\sqrt{2}} A \langle \frac{3}{2}, 1; +\frac{3}{2}, -1 | \frac{1}{2}, +\frac{1}{2} \rangle, \\ \langle 3S, \frac{1}{2}, -\frac{1}{2} | R_{+1}^{(1)} | 3P, \frac{3}{2}, -\frac{3}{2} \rangle &= \frac{1}{\sqrt{2}} A \langle \frac{3}{2}, 1; -\frac{3}{2}, +1 | \frac{1}{2}, -\frac{1}{2} \rangle, \\ \langle 3S, \frac{1}{2}, -\frac{1}{2} | R_0^{(1)} | 3P, \frac{3}{2}, -\frac{1}{2} \rangle &= \frac{1}{\sqrt{2}} A \langle \frac{3}{2}, 1; -\frac{1}{2}, 0 | \frac{1}{2}, -\frac{1}{2} \rangle, \\ \langle 3S, \frac{1}{2}, -\frac{1}{2} | R_{-1}^{(1)} | 3P, \frac{3}{2}, +\frac{1}{2} \rangle &= \frac{1}{\sqrt{2}} A \langle \frac{3}{2}, 1; +\frac{1}{2}, -1 | \frac{1}{2}, -\frac{1}{2} \rangle. \end{aligned}$$

3. An electron is in spin state $|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle + e^{i\phi} |-\rangle)$, but the angle ϕ , is unknown, and uniformly distributed between 0 and π .

(a) [8] Compute the state operator (also known as density matrix) in matrix form. Check that the trace is correct.

The state operator is given by $\rho = \sum_i f_i |\psi_i\rangle\langle\psi_i|$, but in this case we convert it into an integral over ϕ , averaging over the whole range of ϕ , so we have

$$\begin{aligned}\rho &= \frac{1}{\pi} \int_0^\pi |\psi_\phi\rangle\langle\psi_\phi| d\phi = \frac{1}{2\pi} \int_0^\pi \begin{pmatrix} 1 & \\ e^{i\phi} & \end{pmatrix} \begin{pmatrix} 1 & e^{-i\phi} \\ & \end{pmatrix} d\phi = \frac{1}{4\pi} \int_0^\pi \begin{pmatrix} 1 & e^{i\phi} \\ e^{-i\phi} & 1 \end{pmatrix} d\phi \\ &= \frac{1}{2\pi} \begin{pmatrix} \phi & -ie^{i\phi} \\ ie^{-i\phi} & \phi \end{pmatrix} \Big|_0^\pi = \frac{1}{2\pi} \begin{pmatrix} \pi & -i(-1-1) \\ i(-1-1) & \pi \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{\pi}i \\ \frac{1}{\pi}i & \frac{1}{2} \end{pmatrix}.\end{aligned}$$

The trace, of course, is $\text{Tr}(\rho) = \frac{1}{2} + \frac{1}{2} = 1$, which is correct.

(b) [9] Calculate the expectation values of the three spin operators $\langle S \rangle$.

We simply compute each of these straightforwardly:

$$\begin{aligned}\langle S_x \rangle &= \text{Tr}(\rho S_x) = \frac{1}{2} \hbar \text{Tr} \left[\begin{pmatrix} \frac{1}{2} & -\frac{1}{\pi}i \\ \frac{1}{\pi}i & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \frac{1}{2} \hbar \text{Tr} \begin{pmatrix} -\frac{1}{\pi}i & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\pi}i \end{pmatrix} = \frac{1}{2} \hbar 0 = 0, \\ \langle S_y \rangle &= \text{Tr}(\rho S_y) = \frac{1}{2} \hbar \text{Tr} \left[\begin{pmatrix} \frac{1}{2} & -\frac{1}{\pi}i \\ \frac{1}{\pi}i & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] = \frac{1}{2} \hbar \text{Tr} \begin{pmatrix} \frac{1}{\pi} & -\frac{1}{2}i \\ \frac{1}{2}i & \frac{1}{\pi} \end{pmatrix} = \frac{2}{2\pi} \hbar = \frac{\hbar}{\pi}, \\ \langle S_x \rangle &= \text{Tr}(\rho S_x) = \frac{1}{2} \hbar \text{Tr} \left[\begin{pmatrix} \frac{1}{2} & -\frac{1}{\pi}i \\ \frac{1}{\pi}i & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = \frac{1}{2} \hbar \text{Tr} \begin{pmatrix} \frac{1}{2} & \frac{1}{\pi}i \\ \frac{1}{\pi}i & -\frac{1}{2} \end{pmatrix} = \frac{1}{2} \hbar \left(\frac{1}{2} - \frac{1}{2} \right) = 0.\end{aligned}$$

(c) [8] Show that if the Hamiltonian is of the form $H = \hbar\omega\sigma_y$, then ρ is time-independent.

We compute the time derivative of the state operator at $t = 0$ using

$$\begin{aligned}i\hbar \frac{d\rho}{dt} &= [H, \rho] = \hbar\omega [\sigma_y, \rho] = \hbar\omega \left\{ \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{\pi}i \\ \frac{1}{\pi}i & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & -\frac{1}{\pi}i \\ \frac{1}{\pi}i & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right\} \\ &= \hbar\omega \left\{ \begin{pmatrix} \frac{1}{\pi} & -\frac{1}{2}i \\ \frac{1}{2}i & \frac{1}{\pi} \end{pmatrix} - \begin{pmatrix} \frac{1}{\pi} & -\frac{1}{2}i \\ \frac{1}{2}i & \frac{1}{\pi} \end{pmatrix} \right\} = 0.\end{aligned}$$

Since it is unchanging at $t = 0$, it will not change, and hence it will be time-independent.

4. In one of your problem sets, you showed that a spinless particle of mass m in a potential $V(\mathbf{r}) = \frac{1}{2}m\omega^2\mathbf{r}^2$ has eigenstates $|n, l, m\rangle$ with energy $E = \hbar\omega(n + \frac{3}{2})$, and l and m are the quantum numbers corresponding to L^2 and L_z respectively. In *this* problem, an electron (which has spin) of mass m will be influenced by scalar and vector potentials

$$U(\mathbf{R}) = -\frac{1}{2}\alpha\mathbf{R}^2 \quad \text{and} \quad \mathbf{A}(\mathbf{R}) = \frac{1}{2}\beta(X\hat{\mathbf{y}} - Y\hat{\mathbf{x}})$$

- (a) [5] Find the electric and magnetic fields at all points.

The electric and magnetic fields are given by

$$\mathbf{E} = -\nabla U - \partial\mathbf{A}/\partial t = \frac{1}{2}\alpha\nabla(x^2 + y^2 + z^2) = \frac{1}{2}\alpha(2x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 2z\hat{\mathbf{z}}) = \alpha\mathbf{r},$$

$$\mathbf{B} = \nabla \times \mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = \frac{1}{2}\beta\hat{\mathbf{z}}\left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) = \beta\hat{\mathbf{z}}.$$

- (b) [8] For an electron, write down the full Hamiltonian, expanding it out in powers of β . Show that some of the terms proportional to β can be written in terms of L_z .

The full Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2m}(\mathbf{P} + e\mathbf{A})^2 - eU + \frac{ge}{2m}\mathbf{B} \cdot \mathbf{S} \\ &= \frac{1}{2m}\left[(P_x - \frac{1}{2}e\beta Y)^2 + (P_y + \frac{1}{2}e\beta X)^2 + P_z^2\right] + \frac{1}{2}e\alpha\mathbf{R}^2 + \frac{ge\beta}{2m}S_z \\ &= \frac{1}{2m}\left[P_x^2 - e\beta YP_x + \frac{1}{4}e^2\beta^2 Y^2 + P_y^2 + e\beta XP_y + \frac{1}{4}e^2\beta^2 X^2 + P_z^2\right] + \frac{1}{2}e\alpha\mathbf{R}^2 + \frac{ge\beta}{2m}S_z \\ &= \frac{1}{2m}\mathbf{P}^2 + \frac{\beta e}{2m}(XP_y - YP_x) + \frac{\beta^2 e^2}{8m}(X^2 + Y^2) + \frac{1}{2}e\alpha\mathbf{R}^2 + \frac{ge\beta}{2m}S_z. \end{aligned}$$

We then notice that $L_z = XP_y - YP_x$.

- (c) [6] If $\beta = 0$, find the eigenstates and energies of the resulting Hamiltonian.

If $\beta = 0$, the Hamiltonian is just $H = \mathbf{P}^2/(2m) + \frac{1}{2}e\alpha\mathbf{R}^2$, which is the harmonic oscillator. We just equate $\frac{1}{2}m\omega^2\mathbf{R}^2 = \frac{1}{2}e\alpha\mathbf{R}^2$, so $\omega^2 = e\alpha/m$. The states are the same as we found in the homework, namely $|n, l, m\rangle$, except we also have spin, so our states instead will be $|n, l, m, m_s\rangle$. So we have

$$|n, l, m, m_s\rangle \quad \text{with eigenvalues} \quad E = \hbar\omega(n + \frac{3}{2}) = \hbar\sqrt{\frac{e\alpha}{m}}(n + \frac{3}{2}).$$

- (d)[6] Suppose $\beta \neq 0$, but assume β is small enough that the β^2 terms can be dropped. Show that the eigenstates of part (c) are still eigenstates. Find the new energy eigenvalues.

Dropping the β^2 terms, the Hamiltonian is now

$$H = \frac{1}{2m} \mathbf{P}^2 + \frac{1}{2} e\alpha \mathbf{R}^2 + \frac{\beta e}{2m} L_z + \frac{ge\beta}{2m} S_z.$$

The first two terms are identical with before, and have the same energy. The states are also eigenstates of L_z and S_z with eigenvalues of $\hbar m$ and $\hbar m_s$ respectively. Therefore, we have

$$|n, l, m, m_s\rangle \quad \text{with eigenvalues} \quad E = \hbar \sqrt{\frac{e\alpha}{m}} \left(n + \frac{3}{2} \right) + \frac{\hbar e}{2m} (m + gm_s).$$

5. Two spinless non-interacting particles have mass m and are both in an infinite square well with allowed region $0 < x < a$. The one-particle eigenstate wave functions are

$$\psi_n(x) = \sqrt{2/a} \sin(n\pi x/a), \text{ which have energy } E_n = \pi^2 n^2 \hbar^2 / (2ma^2).$$

- (a) [6] If the two particles are non-identical particles, write explicitly the wave function $\psi(x_1, x_2)$ if the first one has $n = 1$ and the second has $n = 2$. What is the energy of this state?

For non-interacting particles, the wave function is just the product, and the energy is the sum of the energies, so we have

$$\psi(x_1, x_2) = \frac{2}{a} \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{2\pi x_2}{a}\right) \quad \text{with energy} \quad E = \frac{\pi^2 \hbar^2}{2ma^2} (1^2 + 2^2) = \frac{5\pi^2 \hbar^2}{2ma^2}.$$

- (b) [9] If the two particles are identical, write explicitly the wave function if one has $n = 1$ and the other has $n = 2$, if they are (i) bosons or (ii) fermions.

For identical particles, the state vector must be symmetrized for bosons and anti-symmetrized for fermions; that is, the state will now be $|\psi\rangle = \frac{1}{\sqrt{2}}(|1, 2\rangle \pm |2, 1\rangle)$, with the plus for bosons and the minus for fermions. This means that the wave function will be

$$\psi(x_1, x_2) = \frac{\sqrt{2}}{a} \left[\sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{2\pi x_2}{a}\right) \pm \sin\left(\frac{\pi x_2}{a}\right) \sin\left(\frac{2\pi x_1}{a}\right) \right].$$

- (c) [10] For each of the three cases (non-identical, bosons, or fermions) find the probability density that both particles are simultaneously at $x = \frac{1}{3}a$. If the particles actually repel each other, so they don't like to be at the same place, which case would probably result in the lowest energy?

To find the probability density of the particles can both be at given points is $|\psi(x_1, x_2)|^2$.

For non-identical particles, we have

$$\psi\left(\frac{1}{3}a, \frac{1}{3}a\right) = \frac{2}{a} \sin\left(\frac{1}{3}\pi\right) \sin\left(\frac{2}{3}\pi\right) = \frac{2}{a} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3}{2a}, \quad \left|\psi\left(\frac{1}{3}a, \frac{1}{3}a\right)\right| = \frac{9}{4a^2}.$$

For the bosons or fermions, we have

$$\begin{aligned} \psi\left(\frac{1}{3}a, \frac{1}{3}a\right) &= \frac{\sqrt{2}}{a} \left[\sin\left(\frac{1}{3}\pi\right) \sin\left(\frac{2}{3}\pi\right) \pm \sin\left(\frac{2}{3}\pi\right) \sin\left(\frac{1}{3}\pi\right) \right] = \frac{\sqrt{2}}{a} \cdot \left[\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \pm \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \right] \\ &= \frac{3}{2a\sqrt{2}} (1 \pm 1) = \begin{cases} \frac{3}{\sqrt{2}} a^{-1} & \text{for bosons,} \\ 0 & \text{for fermions.} \end{cases} \end{aligned}$$

Summarizing all our information, we have

$$\left|\psi\left(\frac{1}{3}a, \frac{1}{3}a\right)\right|^2 = \begin{cases} \frac{9}{4} a^{-2} & \text{for non-identical,} \\ \frac{9}{2} a^{-2} & \text{for bosons,} \\ 0 & \text{for fermions.} \end{cases}$$

If the particles repel each other, they will have the highest energy for bosons and the lowest energy for fermions.

6. Consider N identical non-interacting spin-1/2 particles in a one-dimensional harmonic oscillator with potential $V(x) = \frac{1}{2}m\omega^2 x^2$.

- (a) [3] What are the energy eigenstates for a single particle? If we take spin into account, does this energy change, and how many particles fit into each state?

The harmonic oscillator has energies $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ where $n = 0, 1, 2, \dots$. This isn't changed by spin, but because of spin, there will be *two* particles for each value of n .

- (b) [12] If we have N particles in the ground state, which states will be occupied? Define the Fermi energy as the average of the highest occupied state and the first unoccupied state. What is the Fermi energy? Work this out in both the even and the odd case.

The first N states will be occupied. If N is even, and each value of n holds two particles, the first $\frac{1}{2}N$ values of n will be used, or $n = 0, 1, 2, \dots, \frac{1}{2}N - 1$.

The first *unoccupied* state will have $n = \frac{1}{2}N$, so averaging these energies, we have

$$E_F = \frac{1}{2}(E_{N/2-1} + E_{N/2}) = \frac{1}{2}[\hbar\omega(\frac{1}{2}N - 1 + \frac{1}{2}) + \hbar\omega(\frac{1}{2}N + \frac{1}{2})] = \frac{1}{2}\hbar\omega(\frac{1}{2}N - \frac{1}{2} + \frac{1}{2}N + \frac{1}{2}) = \frac{1}{2}\hbar\omega N.$$

If N is odd, then the first $N - 1$ particles will be put into the states for $n = 0, 1, 2, \dots, \frac{1}{2}(N - 1) - 1$, and the last particle will be put into one of the two spin states for $n = \frac{1}{2}(N - 1)$. Since the highest occupied and lowest unoccupied states have the same energy, the *average* of these energies is just the energy for $n = \frac{1}{2}(N - 1)$, so

$$E_F = \frac{1}{2}(E_{(N-1)/2} + E_{(N-1)/2}) = E_{(N-1)/2} = \hbar\omega[\frac{1}{2}(N - 1) + \frac{1}{2}] = \frac{1}{2}\hbar\omega N.$$

We note in each case that $E_F = \frac{1}{2}\hbar\omega N$.

(c) [10] What is the exact total energy for all the particles? For this part, assume N is even. Write your answer in the form $E_{tot} \propto NE_F$

For the even case, we have to add up the energy of each particle up to $n = \frac{1}{2}N - 1$, remembering to count two spin states for each value of n .

$$\begin{aligned} E_{tot} &= \sum_{n=0}^{\frac{1}{2}N-1} \sum_{\text{spin}} \hbar\omega(n + \frac{1}{2}) = \hbar\omega \sum_{n=0}^{\frac{1}{2}N-1} (2n + 1) = \hbar\omega [2 \cdot \frac{1}{2}(\frac{1}{2}N)(\frac{1}{2}N - 1) + \frac{1}{2}N] = \hbar\omega(\frac{1}{4}N^2 - \frac{1}{2}N + \frac{1}{2}N) \\ &= \frac{1}{4}N^2\hbar\omega = \frac{1}{2}NE_F. \end{aligned}$$

Sums: $\sum_{p=0}^{M-1} 1 = M$, $\sum_{p=0}^{M-1} p = \frac{1}{2}M(M - 1)$, $\sum_{p=0}^{M-1} p^2 = \frac{1}{6}M(M - 1)(2M + 1)$, $\sum_{p=0}^{M-1} p^3 = \frac{1}{4}M^2(M - 1)^2$

Possibly Useful Formulas	
<p>Spin-1/2: $\mathbf{S} = \frac{1}{2}\hbar\boldsymbol{\sigma}$</p> <p>$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.</p>	<p>EM Fields</p> <p>$\mathbf{B} = \nabla \times \mathbf{A}$</p> <p>$\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla U$</p>
<p>State Operator</p> <p>$\rho = \sum_i f_i \psi_i\rangle\langle\psi_i$</p> <p>$i\hbar \frac{d\rho}{dt} = [H, \rho]$</p>	
<p>Wigner-Eckart Theorem</p> <p>$\langle\alpha, j, m T_q^{(k)} \alpha', j', m'\rangle = \frac{1}{\sqrt{2j+1}} \langle\alpha, j T^{(k)} \alpha', j'\rangle \langle j', k; m', q j, m\rangle$</p>	
<p>EM Hamiltonian</p> <p>$H = \frac{\boldsymbol{\pi}^2}{2m} - eU + \frac{ge}{2m} \mathbf{B} \cdot \mathbf{S}$</p>	