Physics 741 – Graduate Quantum Mechanics

# Solutions to Chapter 9

**3. [25] Suppose an electron lies in a region with electric and magnetic fields: $\mathbf{B} = B\hat{\mathbf{z}}$ and $\mathbf{E} = m\omega_0^2 x\hat{\mathbf{x}}/e$.**

**(a) [2] Find the electric potential $U(x)$ such that $\mathbf{E} = -\nabla U(x)$ that could lead to this electric field.**

We need the potential to get the derivative in the *x*-direction to yield $-m\omega_0^2 x/e$, which tells us that the correct choice is $U(x) = -m\omega_0^2 x^2/2e$. This is easily checked.

**(b) [3] The magnetic field is independent of translations in all three dimensions. However, the electrostatic potential is independent of translations in only two of those dimensions. Find a vector potential A with $\mathbf{B} = \nabla \times \mathbf{A}$ which has translation symmetry in the *same* two directions.**

There are always multiple ways to choose to write the vector potential. The electric potential is translation invariant in the *y*- and *z*-directions, so it makes a lot of sense to try to make our vector potential independent of these two coordinates as well. This means when we write $\mathbf{B} = \nabla \times \mathbf{A}$, we're going to need to get the magnetic field from taking derivatives in the *x*-direction. The way the curl works, this will work out if we choose the magnetic field to lie in the *y*-direction, and it isn't hard to see that this works if $\mathbf{A} = Bx\hat{\mathbf{y}}$.

**(c) [4] Write out the Hamiltonian for this system. Eliminate *B* in terms of the cyclotron frequency $\omega_B = eB/m$. What two translation operators commute with this Hamiltonian? What spin operator commutes with this Hamiltonian?**

The Hamiltonian is

$$H = \frac{1}{2m}(\mathbf{P} + e\mathbf{A})^2 - eU + \frac{ge}{2m}\mathbf{B}\cdot\mathbf{S} = \frac{1}{2m}\left[P_x^2 + \left(P_y + eBX\right)^2 + P_z^2\right] + \tfrac{1}{2}m\omega_0^2 X^2 + \frac{ge}{2m}BS_z$$
$$= \frac{1}{2m}\left[P_x^2 + \left(P_y + m\omega_B X\right)^2 + P_z^2\right] + \tfrac{1}{2}m\omega_0^2 X^2 + \tfrac{1}{2}g\omega_B S_z$$

This commutes with $P_y$, $P_z$, and $S_z$. Life is good.

**(d) [3] Write your wave function in the form $\psi(\mathbf{r}) = X(x)Y(y)Z(z)|m_s\rangle$. Based on some of the operators you worked out in part (c), deduce the form of two of the unknown functions.**

Since our wave function commutes with $P_y$ and $P_z$, we can choose it to be eigenstates of two of these operators, and consequently they will look like $Y(y) = e^{ik_y y}$ and $Z(z) = e^{ik_z z}$. These will have eigenvalues $\hbar k_y$ and $\hbar k_z$ under these two operators.

**(e) [3] Replace the various operators by their eigenvalues in the Hamiltonian. The non-constant terms should be identifiable as a shifted harmonic oscillator.**

Replacing the operators by their eigenvalues, the Hamiltonian becomes

$$H = \frac{1}{2m}\left[P_x^2 + \left(\hbar k_y + m\omega_B X\right)^2 + \hbar^2 k_z^2\right] + \tfrac{1}{2}m\omega_0^2 X^2 + \tfrac{1}{2}g\hbar\omega_B m_s$$
$$= \frac{P_x^2}{2m} + \tfrac{1}{2}m\left(\omega_B^2 + \omega_0^2\right)X^2 + \hbar k_y \omega_B X + \frac{\hbar^2 k_y^2}{2m} + \frac{\hbar^2 k_z^2}{2m} + \tfrac{1}{2}g\hbar\omega_B m_s$$

The last few terms are constants, and the rest is simply a shifted harmonic oscillator.

**(f) [4] Make a simple coordinate replacement that shifts it back. If your formulas match mine up to now, they should look like $X = X' - \hbar k_y \omega_B \big/ \left[m\left(\omega_B^2 + \omega_0^2\right)\right]$.**

We try the suggested substitution.

$$H = \frac{P_x^2}{2m} + \tfrac{1}{2}m\left(\omega_B^2 + \omega_0^2\right)\left[X' - \frac{\hbar k_y \omega_B}{m\left(\omega_B^2 + \omega_0^2\right)}\right]^2 + \hbar k_y \omega_B\left[X' - \frac{\hbar k_y \omega_B}{m\left(\omega_B^2 + \omega_0^2\right)}\right] + \frac{\hbar^2\left(k_y^2 + k_z^2\right)}{2m}$$
$$+ \tfrac{1}{2}g\hbar\omega_B m_s$$
$$= \frac{P_x^2}{2m} + \tfrac{1}{2}m\left(\omega_B^2 + \omega_0^2\right)X'^2 - \frac{\hbar^2 k_y^2 \omega_B^2}{2m\left(\omega_B^2 + \omega_0^2\right)} + \frac{\hbar^2 k_y^2}{2m} + \frac{\hbar^2 k_z^2}{2m} + \tfrac{1}{2}g\hbar\omega_B m_s$$

**(g) [3] Find the energies of the Hamiltonian**

The first two terms are simply a Harmonic oscillator, now not shifted, and the energies are just $\hbar\omega\left(n + \tfrac{1}{2}\right)$, where $\omega = \sqrt{\omega_0^2 + \omega_B^2}$. Therefore the energies are in total

$$E = \hbar\sqrt{\omega_B^2 + \omega_0^2}\left(n + \tfrac{1}{2}\right) + \frac{\hbar^2 k_z^2}{2m} + \frac{\hbar^2 k_y^2 \omega_0^2}{2m\left(\omega_B^2 + \omega_0^2\right)} + \tfrac{1}{2}g\hbar\omega_B m_s$$

**(h) [3] Check that they give sensible answers in the two limits when there is no electric field (pure Landau levels) or no magnetic fields (pure harmonic oscillator plus *y*- and *z*-motion).**

If there are no electric fields, then $\omega_0 = 0$, and we have $E = \dfrac{\hbar^2 k_z^2}{2m} + \hbar\omega_B\left(n + \tfrac{1}{2} + \tfrac{1}{2}gm_s\right)$. This is exactly what we would expect. If there are no magnetic fields, then $\omega_B = 0$, and we have $E = \hbar\omega_0\left(n + \tfrac{1}{2}\right) + \hbar^2\left(k_z^2 + k_y^2\right)\big/2m$, which is a harmonic oscillator added to motion in the *y*- and *z*-direction.