Physics 741 – Graduate Quantum Mechanics

Solutions to Chapter 9

9.1 [10] This problem has nothing to do with quantum mechanics. In the presence of a charged plasma, it is possible to create electromagnetic waves that are longitudinal, having electric polarization parallel to the direction of propagation. The fields take the form $\mathbf{E}(\mathbf{r},t)=E_0\hat{\mathbf{z}}\cos(kz-\omega t)$, but with no magnetic field $\mathbf{B}(\mathbf{r},t)=0$.

(a) [3] Show that this electric field (and lack of magnetic field) can be written purely in terms of a vector potential $\mathbf{A}_1(\mathbf{r},t)$, without the use of the scalar potential, so that $U_1(\mathbf{r},t)=0$.

We need to find a vector field such that $\mathbf{E}=-\partial\mathbf{A}_1/\partial t$, since there is no scalar potential. This can simply be integrated to yield

$$\mathbf{A}_1(\mathbf{r},t)=-\int\mathbf{E}(\mathbf{r},t)\,dt=-\hat{\mathbf{z}}E_0\int\cos(kz-\omega t)\,dt=\hat{\mathbf{z}}\,E_0\sin(kz-\omega t)/\omega .$$

This automatically satisfies $\mathbf{E}=-\partial\mathbf{A}_1/\partial t$, and it also satisfies $\mathbf{B}=\nabla\times\mathbf{A}_1=0$, since the only direction A_1 varies is in the *z*-direction. So setting, $U_1=0$, we have a solution.

(b) [3] Show that the same electric fields (and lack of magnetic field) can also be written purely in terms of a scalar potential $U_2(\mathbf{r},t)$, with no vector potential, so that $\mathbf{A}_2(\mathbf{r},t)=0$.

This time, we need to get $\mathbf{E}=-\nabla U_2$. To make the gradient not vanish in the *z*-direction, we need U_2 to vary in the *z*-direction. This suggests a potential of the form $U_2(z,t)$, which must then satisfy $E_z=-\partial U_2/\partial z$. It follows that

$$U_2(\mathbf{r},t)=-\int E_z dz=-E_0\int\cos(kz-\omega t)\,dz=-\frac{E_0}{k}\sin(kz-\omega t).$$

If we choose $\mathbf{A}_2=0$, then this is guaranteed to satisfy $\mathbf{E}=-\nabla U_2$, and obviously then $\mathbf{B}=\nabla\times\mathbf{A}_2=0$.

(c) [4] Show that these two sets of potential, $(\mathbf{A}_1,U_1)$ and $(\mathbf{A}_2,U_2)$, are related by a gauge transformation, and determine explicitly the form of the gauge function $\chi(\mathbf{r},t)$ that relates them.

We need to find a single function $\chi(\mathbf{r},t)$ such that

$$\mathbf{A}_2(\mathbf{r},t)=\mathbf{A}_1(\mathbf{r},t)+\nabla\chi(\mathbf{r},t)\quad\text{and}\quad U_2(\mathbf{r},t)=U_1(\mathbf{r},t)-\frac{\partial}{\partial t}\chi(\mathbf{r},t).$$

Plugging in our specific values for each of these, we need

$$\nabla\chi(\mathbf{r},t) = -\hat{\mathbf{z}}\frac{E_0}{\omega}\sin(kz-\omega t) \quad \text{and} \quad \frac{\partial}{\partial t}\chi(\mathbf{r},t) = \frac{E_0}{k}\sin(kz-\omega t).$$

Clearly, we want $\chi(\mathbf{r},t)$ to depend only on *z* and *t*. We can then obtain this function via integration:

$$\chi = -\int\frac{E_0}{\omega}\sin(kz-\omega t)\,dz = \frac{E_0}{k\omega}\cos(kz-\omega t) \quad \text{or} \quad \chi = \int\frac{E_0}{k}\sin(kz-\omega t)\,dt = \frac{E_0}{k\omega}\cos(kz-\omega t).$$

2. [20] In chapter two, we defined the probability density ρ and probability current j as

$$\rho = \Psi^*\Psi \quad \text{and} \quad \mathbf{j} = \hbar\left(-i\Psi^*\nabla\Psi + i\Psi\nabla\Psi^*\right)/2m = \left(\Psi^*\mathbf{P}\Psi - \Psi\mathbf{P}\Psi^*\right)/2m,$$

and then derived the conservation of probability formula $\partial\rho/\partial t + \nabla\cdot\mathbf{j} = 0$ from Schrödinger's equation (2.1b). However, Schrödinger's equation has just changed into (9.14), and our proof is no longer valid.

(a) [3] Define the modified probability current $\mathbf{j}'$ the same as j, but replacing $\mathbf{P}\to\boldsymbol{\pi}$ defined by $\boldsymbol{\pi}\Psi = (\mathbf{P}+e\mathbf{A})\Psi$ and $\boldsymbol{\pi}\Psi^* = (\mathbf{P}-e\mathbf{A})\Psi^*$. Find an expression for $\mathbf{j}'$.

The new probability current **j** is

$$\mathbf{j}' = \frac{1}{2m}\left[\Psi^*(\mathbf{P}+e\mathbf{A})\Psi - \Psi(\mathbf{P}-e\mathbf{A})\Psi^*\right] = \frac{1}{2m}\left(\Psi^*\mathbf{P}\Psi - \Psi\mathbf{P}\Psi^*\right) + \frac{e}{m}\Psi^*\mathbf{A}\Psi.$$

(b) [6] Which of the quantities ρ, j, and $\mathbf{j}'$ are gauge invariant?

Under a gauge transformation, $\Psi \to \Psi' = \Psi\exp[-ie\chi/\hbar]$, so

$$\rho \to \Psi'^*\Psi' = \Psi^* e^{ie\chi/\hbar}e^{-ie\chi/\hbar}\Psi = \Psi^*\Psi = \rho,$$

$$\begin{aligned}
\mathbf{j} &\to \frac{\hbar}{2m}\left(-i\Psi'^*\nabla\Psi' + i\Psi'\nabla\Psi'^*\right) = \frac{1}{2m}\left[-i\hbar\Psi^* e^{ie\chi/\hbar}\nabla\left(\Psi e^{-ie\chi/\hbar}\right) + i\hbar\Psi e^{-ie\chi/\hbar}\nabla\left(\Psi^* e^{ie\chi/\hbar}\right)\right] \\
&= \frac{1}{2m}\left\{\Psi^* e^{ie\chi/\hbar}\left[-i\hbar(\nabla\Psi) - e\Psi(\nabla\chi)\right]e^{-ie\chi/\hbar} + \Psi e^{-ie\chi/\hbar}\left[i\hbar(\nabla\Psi^*) - e\Psi^*(\nabla\chi)\right]e^{ie\chi/\hbar}\right\} \\
&= \frac{1}{2m}\left(-i\hbar\Psi^*\nabla\Psi + i\hbar\Psi\nabla\Psi^* - 2e\Psi^*\Psi\nabla\chi\right) = \mathbf{j} - \frac{e}{m}\Psi\Psi^*\nabla\chi
\end{aligned}$$

$$\begin{aligned}
\mathbf{j}' &\to \frac{\hbar}{2m}\left(-i\Psi'^*\nabla\Psi' + i\Psi'\nabla\Psi'^*\right) + \frac{e}{m}\Psi'^*\mathbf{A}'\Psi' \\
&= \mathbf{j} - \frac{e}{m}\Psi\Psi^*\nabla\chi + \frac{e}{m}\left(\Psi^* e^{ie\chi/\hbar}\right)(\mathbf{A}+\nabla\chi)\left(\Psi e^{-ie\chi/\hbar}\right) = \mathbf{j} + \frac{e}{m}\Psi^*\mathbf{A}\Psi = \mathbf{j}'.
\end{aligned}$$

Obviously, the probability density ρ and modified current $\mathbf{j}'$ are gauge invariant, but the original current $\mathbf{j}$ is not.

(c) [11] Demonstrate that only one of $\partial\rho/\partial t + \nabla\cdot\mathbf{j} = 0$ and $\partial\rho/\partial t + \nabla\cdot\mathbf{j}' = 0$ is still valid, using the modified Schrödinger's equation (9.15).

We start with Schrödinger's equation. We multiply it on the left by Ψ^* to yield

$$i\hbar\Psi^*\frac{\partial\Psi}{\partial t} = \frac{1}{2m}\Psi^*\left(\mathbf{P}+e\mathbf{A}\right)^2\Psi - eU\Psi^*\Psi = \frac{1}{2m}\Psi^*\left(-i\hbar\nabla+e\mathbf{A}\right)^2\Psi - eU\Psi^*\Psi\,.$$

Taking the complex conjugate of this expression, we have

$$-i\hbar\Psi\frac{\partial\Psi^*}{\partial t} = \frac{1}{2m}\Psi\left(i\hbar\nabla+e\mathbf{A}\right)^2\Psi^* - eU\Psi^*\Psi\Psi^*\,.$$

Subtracting this from the previous equation, we have

$$i\hbar\left(\Psi^*\frac{\partial\Psi}{\partial t}+\Psi\frac{\partial\Psi^*}{\partial t}\right) = \frac{1}{2m}\left[\Psi^*\left(-i\hbar\nabla+e\mathbf{A}\right)^2\Psi - \Psi\left(i\hbar\nabla+e\mathbf{A}\right)^2\Psi^*\right],$$

$$i\hbar\frac{\partial}{\partial t}\left(\Psi^*\Psi\right) = -\frac{\hbar^2}{2m}\left[\Psi^*\nabla^2\Psi-\Psi\nabla^2\Psi^*\right] - \frac{i\hbar e}{2m}\begin{bmatrix}\Psi^*\nabla\cdot\left(\mathbf{A}\Psi\right)+\Psi^*\mathbf{A}\cdot\nabla\Psi \\ +\Psi\nabla\cdot\left(\mathbf{A}\Psi^*\right)+\Psi\mathbf{A}\cdot\nabla\Psi^*\end{bmatrix}.$$

The second set of terms are the new ones in this expression. The first pair can be rewritten with the help of equation (2.18). For the second set, note that the first and fourth expressions can be combined into a single derivative, as can the second and third. The resulting expression is

$$i\hbar\frac{\partial}{\partial t}\rho = -\frac{\hbar^2}{2m}\nabla\cdot\left[\Psi^*\nabla\Psi-\Psi\nabla\Psi^*\right] - \frac{i\hbar e}{2m}\left[\nabla\cdot\left(\Psi^*\mathbf{A}\Psi\right)+\nabla\cdot\left(\Psi\mathbf{A}\Psi^*\right)\right],$$

$$\frac{\partial}{\partial t}\rho = \frac{i\hbar}{2m}\nabla\cdot\left[\Psi^*\nabla\Psi-\Psi\nabla\Psi^*\right] - \frac{e}{m}\nabla\cdot\left(\Psi^*\mathbf{A}\Psi\right)$$

$$= -\nabla\cdot\left[\frac{1}{2m}\left(\Psi^*\mathbf{P}\Psi-\Psi\mathbf{P}\Psi^*\right)+\frac{e}{m}\Psi^*\mathbf{A}\Psi\right] = -\nabla\cdot\mathbf{j}'\,.$$

From here it is obvious that $\partial\rho/\partial t + \nabla\cdot\mathbf{j}' = 0$. It is also clear that $\partial\rho/\partial t + \nabla\cdot\mathbf{j} = 0$ if and only if $\nabla\cdot\left(\Psi^*\mathbf{A}\Psi\right) = 0$, and there is no way to enforce this in general.