

## Solutions to Chapter 9

1. [15] This problem has nothing to do with quantum mechanics. In the presence of a charged plasma, it is possible to create electromagnetic waves that are longitudinal, having electric polarization parallel to the direction of propagation. The fields take the form

$$\mathbf{E}(\mathbf{r}, t) = E_0 \hat{\mathbf{z}} \cos(kz - \omega t), \quad \mathbf{B}(\mathbf{r}, t) = 0.,$$

- (a) [5] Show that this electric field (and lack of magnetic field) can be written purely in terms of a vector potential  $\mathbf{A}_1(\mathbf{r}, t)$ , without the use of the scalar potential, so that  $U_1(\mathbf{r}, t) = 0$ .

We need to find a vector field such that  $\mathbf{E} = -\partial\mathbf{A}_1/\partial t$ , since there is no scalar potential. Since we want  $\mathbf{E}$  to come out looking like a cosine, it makes sense to try letting  $\mathbf{A}_1$  look like a sine. Clearly, the time derivative isn't going to change the direction of  $\mathbf{A}$ , so it makes sense to try expressions like  $\hat{\mathbf{z}} \sin(kz - \omega t)$ . Fiddling a bit, we quickly find the correct solution:

$$\mathbf{A}_1(\mathbf{r}, t) = \hat{\mathbf{z}} \frac{E_0}{\omega} \sin(kz - \omega t) \quad \text{and} \quad U_1(\mathbf{r}, t) = 0.$$

We now simply check that this works

$$\begin{aligned} \mathbf{B}_1(\mathbf{r}, t) &= \nabla \times \mathbf{A}_1(\mathbf{r}, t) = \frac{E_0}{\omega} \nabla \times \hat{\mathbf{z}} \sin(kz - \omega t) = 0, \\ \mathbf{E}_1(\mathbf{r}, t) &= -\frac{E_0}{\omega} \frac{\partial}{\partial t} [\hat{\mathbf{z}} \sin(kz - \omega t)] - \nabla(0) = E_0 \hat{\mathbf{z}} \cos(kz - \omega t). \end{aligned}$$

So it works.

- (b) Show that the same electric fields (and lack of magnetic field) can also be written purely in terms of a scalar potential  $U_2(\mathbf{r}, t)$ , with no vector potential, so that  $\mathbf{A}_2(\mathbf{r}, t) = 0$ .

This time, we need to get  $\mathbf{E} = -\nabla U_1$ . To make the gradient not vanish in the  $z$ -direction, we need  $U_1$  to vary in the  $z$ -direction. Since we are taking a derivative, it again makes sense to try a sine function, something like  $\sin(kz - \omega t)$ . A bit of experimenting then tells you that the correct answer is

$$\mathbf{A}_2(\mathbf{r}, t) = 0 \quad \text{and} \quad U_2(\mathbf{r}, t) = -\frac{E_0}{k} \sin(kz - \omega t).$$

That this is correct is demonstrated simply by testing it

$$\mathbf{B}_1(\mathbf{r}, t) = \nabla \times (0) = 0, \quad \mathbf{E}_1(\mathbf{r}, t) = -\frac{\partial}{\partial t} 0 + \frac{E_0}{k} \nabla \sin(kz - \omega t) = E_0 \hat{\mathbf{z}} \cos(kz - \omega t).$$

**(c) [5] Show that these two sets of potential,  $(\mathbf{A}_1, U_1)$  and  $(\mathbf{A}_2, U_2)$ , are related by a gauge transformation, and determine explicitly the form of the gauge function  $\chi(\mathbf{r}, t)$  that relates them.**

We need to find a single function  $\chi(\mathbf{r}, t)$  such that

$$\mathbf{A}_2(\mathbf{r}, t) = \mathbf{A}_1(\mathbf{r}, t) + \nabla \chi(\mathbf{r}, t) \quad \text{and} \quad U_2(\mathbf{r}, t) = U_1(\mathbf{r}, t) - \frac{\partial}{\partial t} \chi(\mathbf{r}, t)$$

Plugging in our specific values for each of these, we need

$$\nabla \chi(\mathbf{r}, t) = -\hat{\mathbf{z}} \frac{E_0}{\omega} \sin(kz - \omega t) \quad \text{and} \quad \frac{\partial}{\partial t} \chi(\mathbf{r}, t) = \frac{E_0}{k} \sin(kz - \omega t).$$

From the first expression, we see that we want something that varies in the  $z$ -direction, and whose derivative is a sine function, which suggests a cosine. From the second, we see that we want something whose derivative is a sine, which again suggests a cosine. A bit of trial and error will then yield

$$\chi(\mathbf{r}, t) = \frac{E_0}{k\omega} \cos(kz - \omega t).$$

Simple substitution will demonstrate that this works in both equations.

$$\begin{aligned} \nabla \chi(\mathbf{r}, t) &= \frac{E_0}{k\omega} \hat{\mathbf{z}} \frac{\partial}{\partial z} \cos(kz - \omega t) = -\hat{\mathbf{z}} \frac{E_0}{\omega} \sin(kz - \omega t) \\ \frac{\partial}{\partial t} \chi(\mathbf{r}, t) &= \frac{E_0}{k\omega} \frac{\partial}{\partial t} \cos(kz - \omega t) = \frac{E_0}{k} \sin(kz - \omega t) \end{aligned}$$

2. [20] In chapter two, we defined the probability density  $\rho$  and probability current  $\mathbf{j}$  as

$$\rho = \Psi^* \Psi \quad \text{and} \quad \mathbf{j} = \frac{\hbar}{2m} (-i\Psi^* \nabla \Psi + i\Psi \nabla \Psi^*) / 2m = (\Psi^* \mathbf{P} \Psi - \Psi \mathbf{P} \Psi^*) / 2m,$$

and then derived the conservation of probability formula  $\partial \rho / \partial t + \nabla \cdot \mathbf{j} = 0$  from Schrödinger's equation (2.1b). However, Schrödinger's equation has just changed into (9.15), and our proof is no longer valid.

- (a)[3] Define the modified probability current  $\mathbf{j}'$  the same as  $\mathbf{j}$ , but replacing  $\mathbf{P} \rightarrow \boldsymbol{\pi}$  defined by  $\boldsymbol{\pi} \Psi = (\mathbf{P} + e\mathbf{A}) \Psi$  and  $\boldsymbol{\pi} \Psi^* = (\mathbf{P} - e\mathbf{A}) \Psi^*$ . Find an expression for  $\mathbf{j}'$ .

The new probability current  $\mathbf{j}$  is

$$\mathbf{j}' = \frac{1}{2m} [\Psi^* (\mathbf{P} + e\mathbf{A}) \Psi - \Psi (\mathbf{P} - e\mathbf{A}) \Psi^*] = \frac{1}{2m} (\Psi^* \mathbf{P} \Psi - \Psi \mathbf{P} \Psi^*) + \frac{e}{m} \Psi^* \mathbf{A} \Psi.$$

As a matter of practical computation, this can be computed more efficiently as

$$\mathbf{j}' = \text{Re} [\Psi^* (\mathbf{P} + e\mathbf{A}) \Psi] / m.$$

- (b) [6] Which of the quantities  $\rho$ ,  $\mathbf{j}$ , and  $\mathbf{j}'$  are gauge invariant?

Under a gauge transformation,  $\Psi \rightarrow \Psi' = \Psi \exp[-ie\chi/\hbar]$ , so

$$\rho \rightarrow \Psi'^* \Psi' = \Psi^* e^{ie\chi/\hbar} e^{-ie\chi/\hbar} \Psi = \Psi^* \Psi = \rho,$$

$$\begin{aligned} \mathbf{j} &\rightarrow \frac{\hbar}{2m} (-i\Psi'^* \nabla \Psi' + i\Psi' \nabla \Psi'^*) = \frac{1}{2m} [-i\hbar \Psi^* e^{ie\chi/\hbar} \nabla (\Psi e^{-ie\chi/\hbar}) + i\hbar \Psi e^{-ie\chi/\hbar} \nabla (\Psi^* e^{ie\chi/\hbar})] \\ &= \frac{1}{2m} \left\{ \Psi^* e^{ie\chi/\hbar} [-i\hbar (\nabla \Psi) - e\Psi (\nabla \chi)] e^{-ie\chi/\hbar} + \Psi e^{-ie\chi/\hbar} [i\hbar (\nabla \Psi^*) - e\Psi^* (\nabla \chi)] e^{ie\chi/\hbar} \right\} \\ &= \frac{1}{2m} (-i\hbar \Psi^* \nabla \Psi + i\hbar \Psi \nabla \Psi^* - 2e\Psi^* \Psi \nabla \chi) = \mathbf{j} - \frac{e}{m} \Psi \Psi^* \nabla \chi, \end{aligned}$$

$$\begin{aligned} \mathbf{j}' &\rightarrow \frac{\hbar}{2m} (-i\Psi'^* \nabla \Psi' + i\Psi' \nabla \Psi'^*) + \frac{e}{m} \Psi'^* \mathbf{A}' \Psi' \\ &= \mathbf{j} - \frac{e}{m} \Psi \Psi^* \nabla \chi + \frac{e}{m} (\Psi^* e^{ie\chi/\hbar}) (\mathbf{A} + \nabla \chi) (\Psi e^{-ie\chi/\hbar}) = \mathbf{j} + \frac{e}{m} \Psi^* \mathbf{A} \Psi = \mathbf{j}'. \end{aligned}$$

Obviously, the probability density  $\rho$  and modified current  $\mathbf{j}'$  are gauge invariant, but the original current  $\mathbf{j}$  is not.

- (c) [11] Demonstrate that only one of  $\partial \rho / \partial t + \nabla \cdot \mathbf{j} = 0$  and  $\partial \rho / \partial t + \nabla \cdot \mathbf{j}' = 0$  is still valid, using the modified Schrödinger's equation (9.15).

We start with Schrödinger's equation. We multiply it on the left by  $\Psi^*$  to yield

$$i\hbar\Psi^* \frac{\partial\Psi}{\partial t} = \frac{1}{2m}\Psi^* (\mathbf{P} + e\mathbf{A})^2 \Psi - eU\Psi^*\Psi = \frac{1}{2m}\Psi^* (-i\hbar\nabla + e\mathbf{A})^2 \Psi - eU\Psi^*\Psi.$$

Taking the complex conjugate of this expression, we have

$$-i\hbar\Psi \frac{\partial\Psi^*}{\partial t} = \frac{1}{2m}\Psi (i\hbar\nabla + e\mathbf{A})^2 \Psi^* - eU\Psi^*\Psi^*.$$

Subtracting this from the previous equation, we have

$$i\hbar\left(\Psi^* \frac{\partial\Psi}{\partial t} + \Psi \frac{\partial\Psi^*}{\partial t}\right) = \frac{1}{2m}\left[\Psi^* (-i\hbar\nabla + e\mathbf{A})^2 \Psi - \Psi (i\hbar\nabla + e\mathbf{A})^2 \Psi^*\right],$$

$$i\hbar \frac{\partial}{\partial t}(\Psi^*\Psi) = -\frac{\hbar^2}{2m}\left[\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*\right] - \frac{i\hbar e}{2m}\left[\Psi^*\nabla\cdot(\mathbf{A}\Psi) + \Psi^*\mathbf{A}\cdot\nabla\Psi + \Psi\nabla\cdot(\mathbf{A}\Psi^*) + \Psi\mathbf{A}\cdot\nabla\Psi^*\right].$$

The second set of terms are the new ones in this expression. The first pair can be rewritten with the help of equation (2.18). For the second set, note that the first and fourth expressions can be combined into a single derivative, as can the second and third. The resulting expression is

$$i\hbar \frac{\partial}{\partial t} \rho = -\frac{\hbar^2}{2m} \nabla\cdot[\Psi^*\nabla\Psi - \Psi\nabla\Psi^*] - \frac{i\hbar e}{2m} [\nabla\cdot(\Psi^*\mathbf{A}\Psi) + \nabla\cdot(\Psi\mathbf{A}\Psi^*)],$$

$$\frac{\partial}{\partial t} \rho = \frac{i\hbar}{2m} \nabla\cdot[\Psi^*\nabla\Psi - \Psi\nabla\Psi^*] - \frac{e}{m} \nabla\cdot(\Psi^*\mathbf{A}\Psi)$$

$$= -\nabla\cdot\left[\frac{1}{2m}(\Psi^*\mathbf{P}\Psi - \Psi\mathbf{P}\Psi^*) + \frac{e}{m}\Psi^*\mathbf{A}\Psi\right] = -\nabla\cdot\mathbf{j}'.$$

From here it is obvious that  $\partial\rho/\partial t + \nabla\cdot\mathbf{j}' = 0$ . It is also clear that  $\partial\rho/\partial t + \nabla\cdot\mathbf{j} = 0$  if and only if  $\nabla\cdot(\Psi^*\mathbf{A}\Psi) = 0$ , and there is no way to enforce this in general.

### 3. [25] Suppose an electron lies in a region with electric and magnetic fields:

$$\mathbf{B} = B\hat{\mathbf{z}}, \quad \mathbf{E} = m\omega_0^2 x\hat{\mathbf{x}}/e$$

(a) [2] Find the electric potential  $U(x)$  such that  $\mathbf{E} = -\nabla U(x)$  that could lead to this electric field.

We need the potential to get the derivative in the  $x$ -direction to yield  $-m\omega_0^2 x/e$ , which tells us that the correct choice is  $U(x) = -m\omega_0^2 x^2/2e$ . This is easily checked.

- (b) [3] The magnetic field is independent of translations in all three dimensions. However, the electrostatic potential is independent of translations in only two of those dimensions. Find a vector potential  $\mathbf{A}$  with  $\mathbf{B} = \nabla \times \mathbf{A}$  which has translation symmetry in the *same* two directions.

There are always multiple ways to choose to write the vector potential. The electric potential is translation invariant in the  $y$ - and  $z$ -directions, so it makes a lot of sense to try to make our vector potential independent of these two coordinates as well. This means when we write  $\mathbf{B} = \nabla \times \mathbf{A}$ , we're going to need to get the magnetic field from taking derivatives in the  $x$ -direction. The way the curl works, this will work out if we choose the magnetic field to lie in the  $y$ -direction, and it isn't hard to see that this works if  $\mathbf{A} = Bx\hat{y}$ .

- (c) [4] Write out the Hamiltonian for this system. Eliminate  $B$  in terms of the cyclotron frequency  $\omega_B = eB/m$ . What two translation operators commute with this Hamiltonian? What spin operator commutes with this Hamiltonian?

The Hamiltonian is

$$H = \frac{1}{2m}(\mathbf{P} + e\mathbf{A})^2 - eU + \frac{ge}{2m}\mathbf{B} \cdot \mathbf{S} = \frac{1}{2m} \left[ P_x^2 + (P_y + eBX)^2 + P_z^2 \right] + \frac{1}{2}m\omega_0^2 X^2 + \frac{ge}{2m}BS_z$$

$$= \frac{1}{2m} \left[ P_x^2 + (P_y + m\omega_B X)^2 + P_z^2 \right] + \frac{1}{2}m\omega_0^2 X^2 + \frac{1}{2}g\omega_B S_z$$

This commutes with  $P_y$ ,  $P_z$ , and  $S_z$ . Life is good.

- (d) [3] Write your wave function in the form  $\psi(\mathbf{r}) = X(x)Y(y)Z(z)|m_s\rangle$ . Based on some of the operators you worked out in part (c), deduce the form of two of the unknown functions.

Since our wave function commutes with  $P_y$  and  $P_z$ , we can choose it to be eigenstates of two of these operators, and consequently they will look like  $Y(y) = e^{ik_y y}$  and  $Z(z) = e^{ik_z z}$ . These will have eigenvalues  $\hbar k_y$  and  $\hbar k_z$  under these two operators.

- (e) [3] Replace the various operators by their eigenvalues in the Hamiltonian. The non-constant terms should be identifiable as a shifted harmonic oscillator.

Replacing the operators by their eigenvalues, the Hamiltonian becomes

$$\begin{aligned}
H &= \frac{1}{2m} \left[ P_x^2 + (\hbar k_y + m\omega_B X)^2 + \hbar^2 k_z^2 \right] + \frac{1}{2} m\omega_0^2 X^2 + \frac{1}{2} g\hbar\omega_B m_s \\
&= \frac{P_x^2}{2m} + \frac{1}{2} m(\omega_B^2 + \omega_0^2) X^2 + \hbar k_y \omega_B X + \frac{\hbar^2 k_y^2}{2m} + \frac{\hbar^2 k_z^2}{2m} + \frac{1}{2} g\hbar\omega_B m_s
\end{aligned}$$

The last few terms are constants, and the rest is simply a shifted harmonic oscillator.

**(f) [4] Make a simple coordinate replacement that shifts it back. If your formulas match mine up to now, they should look like:**

$$X = X' - \frac{\hbar k_y \omega_B}{m(\omega_B^2 + \omega_0^2)}.$$

We try the suggested substitution.

$$\begin{aligned}
H &= \frac{P_x^2}{2m} + \frac{1}{2} m(\omega_B^2 + \omega_0^2) \left[ X' - \frac{\hbar k_y \omega_B}{m(\omega_B^2 + \omega_0^2)} \right]^2 + \hbar k_y \omega_B \left[ X' - \frac{\hbar k_y \omega_B}{m(\omega_B^2 + \omega_0^2)} \right] + \frac{\hbar^2 (k_y^2 + k_z^2)}{2m} \\
&\quad + \frac{1}{2} g\hbar\omega_B m_s \\
&= \frac{P_x^2}{2m} + \frac{1}{2} m(\omega_B^2 + \omega_0^2) X'^2 - \frac{\hbar^2 k_y^2 \omega_B^2}{2m(\omega_B^2 + \omega_0^2)} + \frac{\hbar^2 k_y^2}{2m} + \frac{\hbar^2 k_z^2}{2m} + \frac{1}{2} g\hbar\omega_B m_s
\end{aligned}$$

**(g) [3] Find the energies of the Hamiltonian**

The first two terms are simply a Harmonic oscillator, now not shifted, and the energies are just  $\hbar\omega(n + \frac{1}{2})$ , where  $\omega = \sqrt{\omega_0^2 + \omega_B^2}$ . Therefore the energies are in total

$$E = \hbar\sqrt{\omega_B^2 + \omega_0^2} \left( n + \frac{1}{2} \right) + \frac{\hbar^2 k_z^2}{2m} + \frac{\hbar^2 k_y^2 \omega_0^2}{2m(\omega_B^2 + \omega_0^2)} + \frac{1}{2} g\hbar\omega_B m_s$$

**(h) [3] Check that they give sensible answers in the two limits when there is no electric field (pure Landau levels) or no magnetic fields (pure harmonic oscillator plus y- and z-motion).**

If there are no electric fields, then  $\omega_0 = 0$ , and we have

$E = \hbar^2 k_z^2 / 2m + \hbar\omega_B (n + \frac{1}{2} + \frac{1}{2} g m_s)$ . This is exactly what we would expect. If there are no magnetic fields, then  $\omega_B = 0$ , and we have  $E = \hbar\omega_0 (n + \frac{1}{2}) + \hbar^2 (k_z^2 + k_y^2) / 2m$ , which is a harmonic oscillator added to motion in the y- and z-direction.