Solutions to Chapter 8

5. [20] Suppose we have two particles with spin ½, and they are in the total spin-0 state, $\left|00\right\rangle = \frac{1}{\sqrt{2}}\left(\left|+_z -_z\right\rangle - \left|-_z +_z\right\rangle\right)$, where we have introduced the subscripts *z* to make clear that the states are eigenstates of the two operators S_{1z} and S_{2z}. Let's define the spin operators $S_{i\theta}$ as $S_{i\theta} = \frac{1}{2}\hbar\left(\cos\theta\sigma_{iz} + \sin\theta\sigma_{ix}\right)$, where *i* = 1, 2 simply denotes which of the two spins we are measuring.

(a) [6] Show that the state can be written in the form $\left|00\right\rangle = \frac{1}{\sqrt{2}}\left(\left|+_\theta -_\theta\right\rangle - \left|-_\theta +_\theta\right\rangle\right)$, where these states are eigenstates of the two operators $S_{i\theta}$.

To make things a bit simpler, I'll drop the *z* subscripts whenever I can. Well, let's follow the advice and see where it ends up

$$\begin{aligned}
\left|+_\theta -_\theta\right\rangle &= \left|+_\theta\right\rangle \otimes \left|-_\theta\right\rangle = \left[\cos\left(\tfrac{1}{2}\theta\right)\left|+\right\rangle + \sin\left(\tfrac{1}{2}\theta\right)\left|-\right\rangle\right] \otimes \left[-\sin\left(\tfrac{1}{2}\theta\right)\left|+\right\rangle + \cos\left(\tfrac{1}{2}\theta\right)\left|-\right\rangle\right] \\
&= -\cos\left(\tfrac{1}{2}\theta\right)\sin\left(\tfrac{1}{2}\theta\right)\left|++\right\rangle + \cos^2\left(\tfrac{1}{2}\theta\right)\left|+-\right\rangle - \sin^2\left(\tfrac{1}{2}\theta\right)\left|-+\right\rangle + \cos\left(\tfrac{1}{2}\theta\right)\sin\left(\tfrac{1}{2}\theta\right)\left|--\right\rangle, \\
\left|-_\theta +_\theta\right\rangle &= \left|-_\theta\right\rangle \otimes \left|+_\theta\right\rangle = \left[-\sin\left(\tfrac{1}{2}\theta\right)\left|+\right\rangle + \cos\left(\tfrac{1}{2}\theta\right)\left|-\right\rangle\right] \otimes \left[\cos\left(\tfrac{1}{2}\theta\right)\left|+\right\rangle + \sin\left(\tfrac{1}{2}\theta\right)\left|-\right\rangle\right] \\
&= -\cos\left(\tfrac{1}{2}\theta\right)\sin\left(\tfrac{1}{2}\theta\right)\left|++\right\rangle - \sin^2\left(\tfrac{1}{2}\theta\right)\left|+-\right\rangle + \cos^2\left(\tfrac{1}{2}\theta\right)\left|-+\right\rangle + \cos\left(\tfrac{1}{2}\theta\right)\sin\left(\tfrac{1}{2}\theta\right)\left|--\right\rangle
\end{aligned}$$

So we have

$$\begin{aligned}
\tfrac{1}{\sqrt{2}}\left(\left|+_\theta -_\theta\right\rangle - \left|-_\theta +_\theta\right\rangle\right) &= \tfrac{1}{\sqrt{2}}\left\{\left[\cos^2\left(\tfrac{1}{2}\theta\right) + \sin^2\left(\tfrac{1}{2}\theta\right)\right]\left|+-\right\rangle - \left[\sin^2\left(\tfrac{1}{2}\theta\right) + \cos^2\left(\tfrac{1}{2}\theta\right)\right]\left|-+\right\rangle\right\} \\
&= \tfrac{1}{\sqrt{2}}\left(\left|+-\right\rangle - \left|-+\right\rangle\right) = \left|00\right\rangle
\end{aligned}$$

(b) [6] Suppose you measure $S_{1\theta}$, the spin of particle 1 at an angle θ. What is the probability that you get each of the possible results? In each case, what is the wave function afterwards?

According to our rules for doing quantum mechanics, we must first find eigenstates of $S_{1\theta}$. One way to do this is to use the states $\left|\pm_\theta \pm_\theta\right\rangle$, the four states that are eigenstates of $S_{1\theta}$ and $S_{2\theta}$. We then have

$$\begin{aligned}
P_1(+) &= \sum_{m_\theta=\pm}\left|\left\langle +_\theta m_\theta \middle| \psi\right\rangle\right|^2 = \sum_{m_\theta=\pm}\left|\tfrac{1}{\sqrt{2}}\left(\left\langle +_\theta m_\theta \middle| +_\theta -_\theta\right\rangle - \left\langle +_\theta m_\theta \middle| -_\theta +_\theta\right\rangle\right)\right|^2 = \tfrac{1}{2}\sum_{m_\theta=\pm}\left|\left\langle +_\theta m_\theta \middle| +_\theta -_\theta\right\rangle\right|^2 = \tfrac{1}{2}, \\
P_1(-) &= \sum_{m_\theta=\pm}\left|\left\langle -_\theta m_\theta \middle| \psi\right\rangle\right|^2 = \sum_{m_\theta=\pm}\left|\tfrac{1}{\sqrt{2}}\left(\left\langle -_\theta m_\theta \middle| +_\theta -_\theta\right\rangle - \left\langle -_\theta m_\theta \middle| -_\theta +_\theta\right\rangle\right)\right|^2 = \tfrac{1}{2}\sum_{m_\theta=\pm}\left|-\left\langle -_\theta m_\theta \middle| -_\theta +_\theta\right\rangle\right|^2 = \tfrac{1}{2},
\end{aligned}$$

so the probabilities are evenly split between the two possibilities.

The state afterwards will depend on what the result of the measurement is. If the measurement yields the value +, then the state vector will be

$$\left|\psi_+\right\rangle = \sum_{m_\theta=\pm} \frac{\left|+_\theta m_\theta\right\rangle\left\langle +_\theta m_\theta \middle| 00\right\rangle}{\sqrt{P_1(+)}} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{\frac{1}{2}}} \sum_{m_\theta=\pm} \left(\left|+_\theta m_\theta\right\rangle\left\langle +_\theta m_\theta \middle| +_\theta -_\theta\right\rangle - \left|+_\theta m_\theta\right\rangle\left\langle +_\theta m_\theta \middle| -_\theta +_\theta\right\rangle\right)$$
$$= \left|+_\theta -_\theta\right\rangle\left\langle +_\theta -_\theta \middle| +_\theta -_\theta\right\rangle = \left|+_\theta -_\theta\right\rangle.$$

If, on the other hand, the measurement yields the value -, then the wave function will be

$$\left|\psi_-\right\rangle = \sum_{m_\theta=\pm} \frac{\left|-_\theta m_\theta\right\rangle\left\langle -_\theta m_\theta \middle| 00\right\rangle}{\sqrt{P_1(-)}} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{\frac{1}{2}}} \sum_{m_\theta=\pm} \left(\left|-_\theta m_\theta\right\rangle\left\langle -_\theta m_\theta \middle| +_\theta -_\theta\right\rangle - \left|-_\theta m_\theta\right\rangle\left\langle -_\theta m_\theta \middle| -_\theta +_\theta\right\rangle\right)$$
$$= -\left|-_\theta +_\theta\right\rangle\left\langle -_\theta +_\theta \middle| -_\theta +_\theta\right\rangle = -\left|-_\theta +_\theta\right\rangle.$$

(c) [7] After completing the measurement in part (b), you perform a *second* measurement, at a different angle, on the other spin, $S_{2\theta'}$. Show in each case that the probability of the second measurement yielding the same result as the first is $P(m_1 = m_2) = \sin^2\left[\frac{1}{2}(\theta' - \theta)\right]$.

This time we need to use eigenstates of $S_{2\theta'}$. The simplest way to complete the problem is to use eigenstates of $S_{1\theta}$ and $S_{2\theta'}$, which look like $\left|\pm_\theta \pm_{\theta'}\right\rangle$. For example, if the first time you get +, then the probability of getting + the second time is

$$P_2(+) = \sum_{m_\theta=\pm} \left|\left\langle m_\theta +_{\theta'} \middle| \psi\right\rangle\right|^2 = \sum_{m_\theta=\pm} \left|\left\langle m_\theta +_{\theta'} \middle| +_\theta -_\theta\right\rangle\right|^2 = \left|\left\langle +_\theta +_{\theta'} \middle| +_\theta -_\theta\right\rangle\right|^2$$

We have only to find the matrix element $\left\langle +_\theta +_{\theta'} \middle| +_\theta -_\theta\right\rangle$ and we'll be done.

An easy way to do this is to write each of them in the $S_{1\theta}$ and S_{2z} basis $\left|\pm_\theta \pm_z\right\rangle$, in terms of which

$$\left|+_\theta -_\theta\right\rangle = \left|+_\theta\right\rangle \otimes \left(-\sin\left(\tfrac{1}{2}\theta\right)\left|+\right\rangle + \cos\left(\tfrac{1}{2}\theta\right)\left|-\right\rangle\right) = -\sin\left(\tfrac{1}{2}\theta\right)\left|+_\theta +\right\rangle + \cos\left(\tfrac{1}{2}\theta\right)\left|+_\theta -\right\rangle$$
$$\left|+_\theta +_{\theta'}\right\rangle = \left|+_\theta\right\rangle \otimes \left(\cos\left(\tfrac{1}{2}\theta'\right)\left|+\right\rangle + \sin\left(\tfrac{1}{2}\theta'\right)\left|-\right\rangle\right) = \cos\left(\tfrac{1}{2}\theta'\right)\left|+_\theta +\right\rangle + \sin\left(\tfrac{1}{2}\theta'\right)\left|+_\theta -\right\rangle$$

So the overlap is

$$\left\langle +_\theta +_{\theta'} \middle| +_\theta -_\theta\right\rangle = \left[\cos\left(\tfrac{1}{2}\theta'\right)\left\langle +_\theta +\right| + \sin\left(\tfrac{1}{2}\theta'\right)\left\langle +_\theta -\right|\right]\left[-\sin\left(\tfrac{1}{2}\theta\right)\left|+\right\rangle + \cos\left(\tfrac{1}{2}\theta\right)\left|-\right\rangle\right]$$
$$= -\cos\left(\tfrac{1}{2}\theta'\right)\sin\left(\tfrac{1}{2}\theta\right) + \sin\left(\tfrac{1}{2}\theta'\right)\cos\left(\tfrac{1}{2}\theta\right) = \sin\left[\tfrac{1}{2}(\theta' - \theta)\right]$$

And our probability in this case is

$$P_2(+) = \sin^2\left[\tfrac{1}{2}(\theta' - \theta)\right]$$

We are half done. We need only complete the other case. Suppose we measured minus on the first measurement. Then the probability of getting minus the second time is

$$P_2(-) = \sum_{m_\theta=\pm} \left|\left\langle m_\theta -_{\theta'} \middle| \psi\right\rangle\right|^2 = \sum_{m_\theta=\pm} \left|-\left\langle m_\theta -_{\theta'} \middle| -_\theta +_\theta\right\rangle\right|^2 = \left|\left\langle -_\theta -_{\theta'} \middle| -_\theta +_\theta\right\rangle\right|^2$$

As before we write

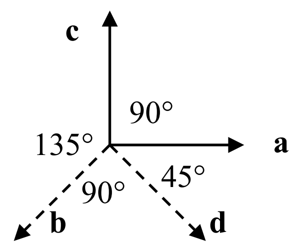
$$\left|-_\theta +_\theta\right\rangle = \left|-_\theta\right\rangle \otimes \left(\cos\left(\tfrac{1}{2}\theta\right)\left|+\right\rangle + \sin\left(\tfrac{1}{2}\theta\right)\left|-\right\rangle\right) = \cos\left(\tfrac{1}{2}\theta\right)\left|-_\theta +\right\rangle + \sin\left(\tfrac{1}{2}\theta\right)\left|-_\theta -\right\rangle$$
$$\left|-_\theta +_{\theta'}\right\rangle = \left|+_\theta\right\rangle \otimes \left(-\sin\left(\tfrac{1}{2}\theta'\right)\left|+\right\rangle + \cos\left(\tfrac{1}{2}\theta'\right)\left|-\right\rangle\right) = -\sin\left(\tfrac{1}{2}\theta'\right)\left|-_\theta +\right\rangle + \cos\left(\tfrac{1}{2}\theta'\right)\left|-_\theta -\right\rangle$$

and our matrix element is

$$\begin{aligned}\left\langle -_\theta -_{\theta'} \middle| -_\theta +_\theta \right\rangle &= \left[\cos\left(\tfrac{1}{2}\theta\right)\left\langle -_\theta +\right| + \sin\left(\tfrac{1}{2}\theta\right)\left\langle -_\theta -\right|\right]\left[-\sin\left(\tfrac{1}{2}\theta'\right)\left|-_\theta +\right\rangle + \cos\left(\tfrac{1}{2}\theta'\right)\left|-_\theta -\right\rangle\right] \\ &= -\cos\left(\tfrac{1}{2}\theta\right)\sin\left(\tfrac{1}{2}\theta'\right) + \sin\left(\tfrac{1}{2}\theta\right)\cos\left(\tfrac{1}{2}\theta'\right) = -\sin\left[\tfrac{1}{2}\left(\theta'-\theta\right)\right], \\ P_2(-) &= \sin^2\left[\tfrac{1}{2}\left(\theta'-\theta\right)\right]\end{aligned}$$

So in both cases, the probability worked out to $P\left(m_1 = m_2\right) = \sin^2\left[\tfrac{1}{2}\left(\theta'-\theta\right)\right]$.

(d) [2] Spin one is going to be measured on one of the two axes a **and** c **sketched at right. Spin two is going to be measured on one of the two axes** b **and** d **sketched at right. What is the probability, as a percent, in all four combinations, that the spins match?**



As shown in the previous calculation, the only thing that matters is the relative angle, so we don't care which axis is the *z*- axis. For the pairs ab, bc, and cd, the angle is 135°, while for ad the angle is 45°, so

$$\begin{aligned}P(a=b) = P(b=c) = P(c=d) &= \sin^2\left[\tfrac{1}{2}135°\right] = 85.36\%, \\ P(a=d) &= \sin^2\left[\tfrac{1}{2}45°\right] = 14.64\%.\end{aligned}$$

6. **[15] Suppose that** U **and** V **are vector operators with respect to some angular momentum operator** J**; that is,** $\left[J_i, U_j\right] = i\hbar\sum_k \varepsilon_{ijk} U_k$ **, and** $\left[J_i, V_j\right] = i\hbar\sum_k \varepsilon_{ijk} V_k$ **. Show that**

   (a) [3] $S = \mathbf{U}\cdot\mathbf{V}$ **is a scalar operator**

We need to prove that S commutes with **J**.

$$\left[J_i, \mathbf{U}\cdot\mathbf{V}\right] = \sum_j \left[J_i, U_j V_j\right] = \sum_j \left\{\left[J_i, U_j\right]V_j + U_j\left[J_i, V_j\right]\right\} = i\hbar\sum_{j,k}\left\{\varepsilon_{ijk}U_k V_j + \varepsilon_{ijk}U_j V_k\right\}$$

Now, rename the dummy indices j and k as k and j in the last term, and then use the fact that ε_{ijk} is anti-symmetric under the interchange of its last two indices.

$$\left[J_i, \mathbf{U}\cdot\mathbf{V}\right] = i\hbar\sum_{j,k} U_k V_j\left\{\varepsilon_{ijk} + \varepsilon_{ikj}\right\} = 0$$

   (b) [12] $\mathbf{W} = \mathbf{U}\times\mathbf{V}$ **is a vector operator**

Although this can be done with the Levi-Civita symbol, it isn't that much harder just to work out all nine cases by hand.

$$\begin{aligned}
\left[J_x, W_x\right] &= \left[J_x, U_y V_z - U_z V_y\right] = U_y\left[J_x, V_z\right] + \left[J_x, U_y\right]V_z - U_z\left[J_x, V_y\right] - \left[J_x, U_z\right]V_y \\
&= i\hbar\left\{-U_y V_y + U_z V_z - U_z V_z + U_y V_y\right\} = 0, \\
\left[J_x, W_y\right] &= \left[J_x, U_z V_x - U_x V_z\right] = \left[J_x, U_z\right]V_x - U_x\left[J_x, V_z\right] = i\hbar\left\{-U_y V_x + U_x V_y\right\} = i\hbar W_z, \\
\left[J_x, W_z\right] &= \left[J_x, U_x V_y - U_y V_x\right] = U_x\left[J_x, V_y\right] - \left[J_x, U_y\right]V_x = i\hbar\left\{U_x V_z - U_z V_x\right\} = -i\hbar W_y, \\
\left[J_y, W_x\right] &= \left[J_y, U_y V_z - U_z V_y\right] = U_y\left[J_y, V_z\right] - \left[J_y, U_z\right]V_y = i\hbar\left\{U_y V_x - U_x V_y\right\} = -i\hbar W_z, \\
\left[J_y, W_y\right] &= \left[J_y, U_z V_x - U_x V_z\right] = U_z\left[J_y, V_x\right] + \left[J_y, U_z\right]V_x - U_x\left[J_y, V_z\right] - \left[J_y, U_x\right]V_z \\
&= i\hbar\left\{-U_z V_z + U_x V_x - U_x V_x + U_z V_z\right\} = 0, \\
\left[J_y, W_z\right] &= \left[J_y, U_x V_y - U_y V_x\right] = \left[J_y, U_x\right]V_y - U_y\left[J_y, V_x\right] = i\hbar\left\{-U_z V_y + U_y V_z\right\} = i\hbar W_x, \\
\left[J_z, W_x\right] &= \left[J_z, U_y V_z - U_z V_y\right] = \left[J_z, U_y\right]V_z - U_z\left[J_z, V_y\right] = i\hbar\left\{-U_x V_z + U_z V_x\right\} = -i\hbar W_y, \\
\left[J_z, W_y\right] &= \left[J_z, U_z V_x - U_x V_z\right] = U_z\left[J_z, V_x\right] - \left[J_z, U_x\right]V_z = i\hbar\left\{U_z V_y - U_y V_z\right\} = i\hbar W_x, \\
\left[J_z, W_z\right] &= \left[J_z, U_x V_y - U_y V_x\right] = U_x\left[J_z, V_y\right] + \left[J_z, U_x\right]V_y - U_y\left[J_z, V_x\right] - \left[J_z, U_y\right]V_x \\
&= i\hbar\left\{-U_x V_x + U_y V_y - U_y V_y + U_x V_x\right\} = 0.
\end{aligned}$$