Physics 741 – Graduate Quantum Mechanics 1

# Solutions to Chapter 8

**2. [10] Suppose that L and S are two sets of angular momentum-like operators that commute with each other, so that $\left[L_i,S_j\right]=0$. In this problem, you may assume from the commutation relations that it follows that $\left[\mathbf{L}^2,\mathbf{L}\right]=0=\left[\mathbf{S}^2,\mathbf{S}\right]$.**

**(a) [2] Define $\mathbf{J}=\mathbf{L}+\mathbf{S}$. Show that $\mathbf{J}$ is also an angular momentum-like operator. It follows automatically that $\left[\mathbf{J}^2,\mathbf{J}\right]=0$.**

We simply have to work out the commutator of each component of **J** with each other. We'll take advantage of the Levi-Civita tensor to show that

$$\left[J_i,J_j\right]=\left[L_i+S_i,L_j+S_j\right]=\left[L_i,L_j\right]+\left[S_i,S_j\right]=i\hbar\sum_k\varepsilon_{ijk}L_k+i\hbar\sum_k\varepsilon_{ijk}S_k=i\hbar\sum_k\varepsilon_{ijk}J_k.$$

This saved us the work of doing it three times.

**(b) [2] Show that $\left[\mathbf{L}^2,\mathbf{J}\right]=0=\left[\mathbf{S}^2,\mathbf{J}\right]$.**

This is really pretty trivial.

$$\left[\mathbf{L}^2,\mathbf{J}\right]=\left[\mathbf{L}^2,\mathbf{L}\right]+\left[\mathbf{L}^2,\mathbf{S}\right]=0+0=0,\quad \left[\mathbf{S}^2,\mathbf{J}\right]=\left[\mathbf{S}^2,\mathbf{L}\right]+\left[\mathbf{S}^2,\mathbf{S}\right]=0+0=0.$$

**(c) [3] Convince yourself (and me) that the four operators $\mathbf{J}^2$, $\mathbf{L}^2$, $\mathbf{S}^2$, and $J_z$ all commute with each other (this is six commutators in all).**

Since $\mathbf{L}^2$ and $\mathbf{S}^2$ commute with **J**, it follows automatically that they commute with $\mathbf{J}^2$ and $J_z$ (that's four commutators so far). Since all the **L**'s and **S**'s commute with each other, it follows that $\mathbf{L}^2$ and $\mathbf{S}^2$ commute with each other. Finally, as mentioned in part (a), since $\left[\mathbf{J}^2,\mathbf{J}\right]=0$, $\mathbf{J}^2$ commutes with $J_z$.

**(d) [3] Convince yourself that $L_z$ and $S_z$ do <u>not</u> commute with $\mathbf{J}^2$.**

We simply try to do the commutation relations and see if it works.

$$\left[\mathbf{J}^2,L_z\right]=\left[\left(\mathbf{L}+\mathbf{S}\right)^2,L_z\right]=\left[\mathbf{L}^2+2\mathbf{L}\cdot\mathbf{S}+\mathbf{S}^2,L_z\right]=\left[\mathbf{L}^2,L_z\right]+2\left[\mathbf{L}\cdot\mathbf{S},L_z\right]+\left[\mathbf{S}^2,L_z\right]$$
$$=2\left[L_x,L_z\right]S_x+2\left[L_y,L_z\right]S_y+2\left[L_z,L_z\right]S_z=-2i\hbar L_yS_x+2i\hbar L_xS_y\ ,$$
$$\left[\mathbf{J}^2,S_z\right]=\left[\left(\mathbf{L}+\mathbf{S}\right)^2,S_z\right]=\left[\mathbf{L}^2+2\mathbf{L}\cdot\mathbf{S}+\mathbf{S}^2,S_z\right]=\left[\mathbf{L}^2,S_z\right]+2\left[\mathbf{L}\cdot\mathbf{S},S_z\right]+\left[\mathbf{S}^2,S_z\right]$$
$$=2L_x\left[S_x,S_z\right]+2L_y\left[S_y,S_z\right]+2L_z\left[S_z,S_z\right]=-2i\hbar L_xS_y+2i\hbar L_yS_x\ .$$

Not surprisingly, the sum of these two expressions is zero, which follows from $\left[\mathbf{J}^2,\mathbf{J}\right]=0$.