Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 8

1. [10] In section A, we were searching for matrices $D(\mathcal{R})$ which satisfy $D(\mathcal{R}_1)D(\mathcal{R}_2)=D(\mathcal{R}_1\mathcal{R}_2)$. One easy way to make this equation work out is to define $D(\mathcal{R})=\mathcal{R}$. Our goal in this problem is to identify the spin.

(a) [4] Using the equations (6.12) and the definition of the spin matrices $D(\mathcal{R}(\hat{\mathbf{r}},\theta))=1-i\theta\hat{\mathbf{r}}\cdot\mathbf{S}/\hbar+\mathcal{O}(\theta^2)$, work out the three spin matrices S.

Equations (6.12) give the rotation matrices around each of the three axes for arbitrary angle θ. If we write these to order θ, we see that they give

$$\mathcal{R}(\hat{\mathbf{x}},\theta)=\begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix}\approx\begin{pmatrix}1&0&0\\0&1&-\theta\\0&\theta&1\end{pmatrix}=\mathbf{1}+\theta\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix},$$

$$\mathcal{R}(\hat{\mathbf{y}},\theta)=\begin{pmatrix}\cos\theta&0&\sin\theta\\0&1&0\\-\sin\theta&0&\cos\theta\end{pmatrix}\approx\begin{pmatrix}1&0&\theta\\0&1&0\\-\theta&0&1\end{pmatrix}=\mathbf{1}+\theta\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},$$

$$\mathcal{R}(\hat{\mathbf{z}},\theta)=\begin{pmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{pmatrix}\approx\begin{pmatrix}1&-\theta&0\\\theta&1&0\\0&0&1\end{pmatrix}=\mathbf{1}+\theta\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}.$$

To linear order, this must be the same as $\mathcal{R}(\hat{\mathbf{r}},\theta)=1-i\theta\hat{\mathbf{r}}\cdot\mathbf{S}/\hbar$, or solving for $\hat{\mathbf{r}}\cdot\mathbf{S}$,

$$S_x=\hbar\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\quad S_y=\hbar\begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix},\quad S_z=\hbar\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}.$$

(b) [3] Find the eigenvalues of S_z. This should be enough for you to conjecture what value of s this representation corresponds to.

The easiest way to find these is to use the characteristic equation, which is

$$0=\det(S_z-\lambda\mathbf{1})=\det\begin{pmatrix}-\lambda&-i\hbar&0\\i\hbar&-\lambda&0\\0&0&-\lambda\end{pmatrix}=-\lambda^3+\hbar^2\lambda=-\lambda(\lambda-\hbar)(\lambda+\hbar)$$

The three solutions of this are $\lambda=0,\hbar,-\hbar$, which are the three values we would expect for spin 1. So we suspect $s=1$. We don't recognize it in this form, because we have not written it in the basis where S_z is diagonalized.

(c) [3] Check explicitly that S^2 is a constant matrix with the appropriate value.

$$\mathbf{S}^2 = \hbar^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}^2 + \hbar^2 \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}^2 + \hbar^2 \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = \hbar^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The result is supposed to be $\mathbf{S}^2 = \left(s^2 + s\right)\hbar^2 = 2\hbar^2$, so it worked out.