

Solutions to Chapter 6

1. [10] For the finite square well in section C, we showed that (6.24) is satisfied for the even wave functions. Repeat this derivation for the odd wave functions; *i.e.*, derive (6.25).

We know from class notes that in the three regions, the solution takes the form

$$\begin{aligned}\psi_I(x) &= Ae^{\beta x} \\ \psi_{II}(x) &= B \cos(kx) + C \sin(kx) \\ \psi_{III}(x) &= De^{-\beta x}\end{aligned}$$

We are interested in the odd parity bound states. Since parity relates regions I and III to each other, and region II to itself, this implies $\psi_I(x) = \psi_{III}(-x)$ and $\psi_{II}(x) = \psi_{II}(-x)$. We therefore have $A = -D$ and $B = 0$.

We now wish to match boundary conditions. We will choose to do so at $x = a$, where we follow the notes to yield

$$\begin{aligned}\psi_{II}(a) = \psi_{III}(a) &\Rightarrow C \sin(ka) = D \exp(-\beta a) \\ \psi'_{II}(a) = \psi'_{III}(a) &\Rightarrow kC \cos(ka) = -\beta D \exp(-\beta a)\end{aligned}$$

Dividing the second equation by the first, we find

$$k \cot(ka) = -\beta$$

Using equations (6.22) and (6.23), it is easy to see that

$$k^2 + \beta^2 = 2mV_0\hbar^{-2} \Rightarrow \beta = \sqrt{2mV_0\hbar^{-2} - k^2}$$

Plugging this in and dividing by $-k$, we find

$$-\cot(ka) = \sqrt{\frac{2mV_0}{\hbar^2 k^2} - 1}$$

This is equation (6.25). Solutions of this equation can then be substituted into (6.23) to get the energy eigenvalues.

2. [15] A particle of mass m in two dimensions is governed by the Hamiltonian

$$H = \frac{1}{2m}(P_x^2 + P_y^2) + \frac{1}{4}\alpha(X^2 + Y^2)^2 + \frac{1}{3}\gamma(X^3 - 3XY^2)$$

(a) [5] Show that the Hamiltonian is invariant under the transformation $R(\mathcal{R}(120^\circ))$.

Under a rotation by 120° , the coordinates transform as

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} \cos \frac{2\pi}{3} & -\sin \frac{2\pi}{3} \\ \sin \frac{2\pi}{3} & \cos \frac{2\pi}{3} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}X - \frac{\sqrt{3}}{2}Y \\ \frac{\sqrt{3}}{2}X - \frac{1}{2}Y \end{pmatrix}$$

We simply substitute this into our potential and check if it remains unchanged. We find

$$\begin{aligned} X^2 + Y^2 &\rightarrow \left(\frac{1}{2}X - \frac{\sqrt{3}}{2}Y\right)^2 + \left(\frac{\sqrt{3}}{2}X + \frac{1}{2}Y\right)^2 = \left(\frac{1}{4}X^2 - \frac{\sqrt{3}}{2}XY + \frac{3}{4}Y^2\right) + \left(\frac{3}{4}X^2 + \frac{\sqrt{3}}{2}XY + \frac{1}{4}Y^2\right) \\ &= X^2 + Y^2 \end{aligned}$$

So this term is unchanged. The other term is more complicated, but with a little work, we see that

$$\begin{aligned} X'^3 - 3X'Y'^2 &= \left(-\frac{1}{2}X - \frac{\sqrt{3}}{2}Y\right)^3 - 3\left(-\frac{1}{2}X - \frac{\sqrt{3}}{2}Y\right)\left(\frac{\sqrt{3}}{2}X - \frac{1}{2}Y\right)^2 \\ &= \left(-\frac{1}{8}X^3 - \frac{3\sqrt{3}}{8}X^2Y - \frac{9}{8}XY^2 - \frac{3\sqrt{3}}{8}Y^3\right) + \left(\frac{3}{2}X + \frac{3\sqrt{3}}{2}Y\right)\left(\frac{3}{4}X^2 - \frac{\sqrt{3}}{2}XY + \frac{1}{4}Y^2\right) \\ &= \left(-\frac{1}{8}X^3 - \frac{3\sqrt{3}}{8}X^2Y - \frac{9}{8}XY^2 - \frac{3\sqrt{3}}{8}Y^3\right) + \left(\frac{9}{8}X^3 + \frac{3\sqrt{3}}{8}X^2Y - \frac{15}{8}XY^2 + \frac{3\sqrt{3}}{8}Y^3\right) \\ &= X^3 - 3XY^2 \end{aligned}$$

Once again, this expression is unchanged, so we have proven our claim that this is unchanged under such a transformation. Hence this potential is invariant under rotations by 120° .

(b) [4] Classify the states according to their eigenvalues under $R(\mathcal{R}(120^\circ))$.

What eigenvalues are possible?

Because the operator $R(\mathcal{R}(120^\circ))$ commutes with the Hamiltonian, our eigenstates of the Hamiltonian can be chosen to also be eigenstates of $R(\mathcal{R}(120^\circ))$. If we define

$$R(\mathcal{R}(120^\circ))|\psi\rangle = \lambda|\psi\rangle,$$

then as always since we have a unitary operator, λ must be a complex number of magnitude one. However, it is further restricted since three successive rotations are identical with no rotation, so we have

$$\lambda^3|\psi\rangle = [R(\mathcal{R}(120^\circ))]^3|\psi\rangle = R(\mathcal{R}(360^\circ))|\psi\rangle = R(1)|\psi\rangle = |\psi\rangle$$

So we have $\lambda^3 = 1$. We can find the three roots in a variety of ways, the easiest being to factor it and use the quadratic equation:

$$\lambda^3 = 1 \Rightarrow 0 = \lambda^3 - 1 = (\lambda - 1)(\lambda^2 + \lambda + 1) \Rightarrow \lambda = 1 \text{ or } \lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

(c) [3] Suppose that $\psi(x, y)$ is an eigenstate of H and $R(\mathcal{R}(120^\circ))$ with eigenvalues E and λ respectively. Show that $\psi^*(x, y)$ is also an eigenstate of H and $R(\mathcal{R}(120^\circ))$, and determine its eigenvalues. (E is, of course, real).

Working in the coordinate representation, Schrödinger's equation and our symmetry relationship are

$$\begin{aligned} -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) &= E\psi(\mathbf{r}) \\ \psi(\mathcal{R}(120^\circ)\mathbf{r}) &= \lambda\psi(\mathbf{r}) \end{aligned}$$

Taking the complex conjugate of these relations, we see that

$$\begin{aligned} -\frac{\hbar^2}{2m}\nabla^2\psi^*(\mathbf{r}) + V(\mathbf{r})\psi^*(\mathbf{r}) &= E\psi^*(\mathbf{r}) \\ \psi^*(\mathcal{R}(120^\circ)\mathbf{r}) &= \lambda^*\psi^*(\mathbf{r}) \end{aligned}$$

In other words, the complex conjugate is also an eigenstate of H and R with eigenvalues E and λ^* respectively.

(d) [3] Careful measurements of the Hamiltonian discovers that the system has some non-degenerate eigenstates (like the ground state), and some states that are two-fold degenerate (two eigenstates with exactly the same eigenvalue). Explain why these degeneracies are occurring.

Any state that has a complex value of λ must come with another state that has eigenvalue λ^* . This will result in two-fold degeneracies. The non-degenerate states correspond to when $\lambda = 1$.

3. [20] A particle of mass m and energy E in two dimensions is incident on a plane step function given by

$$V(X, Y) = \begin{cases} 0 & \text{if } X < 0, \\ V_0 & \text{if } X > 0. \end{cases}$$

The incoming wave has wave function $\psi_{\text{in}}(x, y) = e^{i(k_x x + k_y y)}$ for $x < 0$.

- (a) [7] Write the Hamiltonian. Determine the energy E for the incident wave. Convince yourself that the Hamiltonian has a translation symmetry, and therefore that the transmitted and reflected wave will share something in common with the incident wave (they are all eigenstates of what operator?).

The Hamiltonian, of course, is just $H = P_x^2/2m + P_y^2/2m + V(X)$. Because the potential vanishes for the incoming wave, we have

$$H\psi_{\text{in}} = \frac{P_x^2 + P_y^2}{2m}\psi_{\text{in}} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi_{\text{in}} = \frac{\hbar^2(k_x^2 + k_y^2)}{2m}\psi_{\text{in}},$$

so that $E = \hbar^2(k_x^2 + k_y^2)/2m$. It is obvious that this Hamiltonian commutes with P_y , since the potential doesn't contain Y , and the kinetic part contains only momentum. It follows that eigenstates of the Hamiltonian can be chosen to also be eigenstates of P_y , and will have the same eigenvalue as the incoming wave $P_y|\psi\rangle = \hbar k_y|\psi\rangle$. This tells us the y -dependence will be the same for the incident, reflected, and transmitted wave. The eigenstates will therefore take the form

$$\psi(x, y) = e^{ik_y y} \chi(x).$$

It remains only to find the function $\chi(x)$.

- (b) [7] Write the general form of the reflected and transmitted wave. Use Schrödinger's equation to solve for the values of the unknown parts of the momentum for each of these waves (assume $k_x^2 \hbar^2/2m > V_0$).

If we plug our general solution into Schrödinger's equation, we have

$$E e^{ik_y y} \chi(x) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(e^{ik_y y} \chi(x)\right) + V(x) e^{ik_y y} \chi(x),$$

$$\frac{\hbar^2(k_x^2 + k_y^2)}{2m} e^{ik_y y} \chi(x) = \left(\frac{\hbar^2 k_y^2}{2m} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}\right) e^{ik_y y} \chi(x) + V(x) e^{ik_y y} \chi(x).$$

The exponentials cancel everywhere, and some factors can be cancelled. Solving for the second derivative term, we find

$$\frac{\partial^2 \chi(x)}{\partial x^2} = \left[\frac{2mV(x, y)}{\hbar^2} - k_x^2 \right] \chi(x).$$

This equation is easy to solve. For $x < 0$, the potential is zero, and we are solving $\chi'' = -k^2 \chi$ and the solutions are proportional to $e^{\pm ik_x x}$. For $x > 0$, we can define $k_x'^2 = k_x^2 - 2mV_0/\hbar^2$, and we are solving $\chi'' = -k'^2 \chi$ and the solutions are proportional to $e^{\pm ik'_x x}$.

The incident wave is $\chi_I = e^{+ik_x x}$. The wave going the other way in this region must be the reflected wave, $\chi_R \propto e^{-ik_x x}$. In the other region, the wave proportional to $e^{+ik'_x x}$ is a wave traveling to the right, so we have $\chi_T \propto e^{+ik'_x x}$. The wave proportional to $e^{-ik'_x x}$ would represent a wave moving to the left from infinity on the right, so this doesn't represent anything in this problem. Reinstating the y -dependence, and throwing in some constants of proportionality, our waves are

$$\psi_I = Ae^{+ik_x x + ik_y y}, \quad \psi_R = Be^{-ik_x x + ik_y y}, \quad \psi_T = Ce^{ik'_x x + ik_y y}.$$

(c) [6] Assume the wave function and its derivative are continuous across the boundary $x = 0$. Find the amplitudes for the transmitted and reflected waves, and find the probability R of the wave being reflected.

The wave in the region $x < 0$ is given by $\psi_I + \psi_R$, and on the right, by ψ_T . Matching these wave functions and their derivatives in the x -direction at the boundary, we have

$$Ae^{ik_y y} + Be^{ik_y y} = Ce^{ik_y y} \quad \text{and} \quad Ak_x e^{ik_y y} - Bk_x e^{ik_y y} = Ck'_x e^{ik_y y}$$

When we cancel the common phase on both sides of each of these equations, the first equation becomes $A + B = C$, and substituting this into the second yields

$$Ak_x - Bk_x = (A + B)k'_x.$$

Rearranging this a bit, we have

$$A(k_x - k'_x) = B(k_x + k'_x), \quad B = \frac{k_x - k'_x}{k_x + k'_x} A \quad \text{and} \quad C = A + B = \frac{2k_x}{k_x + k'_x} A$$

The probability of reflection is the ratio of the amplitude squared for the reflected wave vs. the incident wave, so

$$R = \frac{|B|^2}{|A|^2} = \left(\frac{k_x - k'_x}{k_x + k'_x} \right)^2 = \left(\frac{k_x - \sqrt{k_x^2 - 2mV_0/\hbar^2}}{k_x + \sqrt{k_x^2 - 2mV_0/\hbar^2}} \right)^2.$$

4. [15] A particle of mass M in *three* dimensions has potential

$$V(X, Y, Z) = \frac{1}{4} A (X^2 + Y^2)^2.$$

(a) [6] Show that this Hamiltonian has *two* continuous symmetries, and that they commute. Call the corresponding eigenvalues m and k . Are there any restrictions on k and m ?

First, it is obvious that the potential is independent of Z , and therefore there is a continuous translation symmetry in this direction. Secondly, it is easy to see that rotation about the z -axis leaves the Hamiltonian unchanged. Specifically, define a set of rotated operators

$$X' = X \cos \theta - Y \sin \theta$$

$$Y' = X \sin \theta + Y \cos \theta$$

Then if we treat the potential as $V(x, y) = \frac{1}{4} A (x^2 + y^2)^2$, then we have

$$\begin{aligned} V(X', Y') &= \frac{1}{4} A \left[(X \cos \theta - Y \sin \theta)^2 + (X \sin \theta + Y \cos \theta)^2 \right]^2 \\ &= \frac{1}{4} A \left[\begin{array}{l} X^2 \cos^2 \theta - 2XY \cos \theta \sin \theta + Y^2 \sin^2 \theta \\ + X^2 \sin^2 \theta + 2XY \sin \theta \cos \theta + Y^2 \cos^2 \theta \end{array} \right] = \frac{1}{4} A (X^2 + Y^2)^2 = V(X, Y) \end{aligned}$$

Because we have translation symmetry in the z -direction and rotation about the z -axis, our Hamiltonian will commute with the generators of these groups, P_z and L_z . Our energy eigenstates can also be chosen to be eigenstates of these operators, and we will have

$$P_z |\phi\rangle = \hbar k |\phi\rangle \quad \text{and} \quad L_z |\phi\rangle = \hbar m |\phi\rangle$$

As argued in class, the eigenvalue m is forced to be an integer.

(b) [9] What would be an appropriate set of coordinates for writing the eigenstates of this Hamiltonian? Write the eigenstates as a product of three functions (which I call Z , R , and Φ), and give me the explicit form of two of these functions.

Clearly, z is a good coordinate to use, since our eigenstates of the Hamiltonian are eigenstates of P_z . However, since they are also eigenstates of L_z , it seems like a good idea to change coordinates to cylindrical coordinates (ρ, ϕ, z) , which are related to Cartesian coordinates by

$$\left\{ \begin{array}{l} x = \rho \cos \phi \\ y = \rho \sin \phi \\ z = z \end{array} \right\} \quad \text{OR} \quad \left\{ \begin{array}{l} \rho = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{array} \right\}$$

If we write our wave function in terms of these coordinates, and assume it factors, we have

$$\psi(\rho, \phi, z) = R(\rho)\Phi(\phi)Z(z)$$

If we demand that this be an eigenstate of P_z with eigenvalue $\hbar k$, then we find

$$\hbar k Z(z) = P_z Z(z) = \frac{\hbar}{i} \frac{\partial}{\partial z} Z(z) \quad \text{so that} \quad Z(z) = e^{ikz}$$

Similarly, if we demand that $\psi(\rho, \phi, z)$ be an eigenstate of L_z with eigenvalue $\hbar m$, then we find

$$\hbar m \Phi(\phi) = L_z \Phi(\phi) = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \Phi(\phi) \quad \text{so that} \quad \Phi(\phi) = e^{im\phi}$$

There is a certain arbitrariness in normalization, and the choices we have made have perhaps not been the best, but up to a constant, we therefore find

$$\psi(\rho, \phi, z) = R(\rho) e^{ikz + im\phi}.$$

If we wished, we could now easily write an explicit equation for the radial function R . Writing the Laplacian that is implicit in the kinetic term in the Hamiltonian in cylindrical coordinates, we find

$$H\psi = -\frac{\hbar^2}{2M} \left(\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} \right) + \frac{1}{4} A(\rho^2)^2 \psi$$

Plugging in our explicit form for the wave function, and using Schrödinger's equation $H\psi = E\psi$, we have

$$ER = -\frac{\hbar^2}{2M} \left(\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} \right) + \left[\frac{\hbar^2 k^2}{2M} + \frac{\hbar^2 m^2}{2M \rho^2} + \frac{1}{4} A \rho^4 \right] R.$$