

Solutions to Chapter 4

1. [25] It is sometimes said that a watched kettle never boils. In some sense, this is true in quantum mechanics. Consider a quantum system where the state space is two dimensional, with basis states $\{|0\rangle, |1\rangle\}$, the former representing the kettle in the “not boiled” state, the latter the “boiled” state. In terms of these, the Hamiltonian is given by

$$H = \hbar\omega \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

- (a) [6] At $t = 0$, the quantum state is given by $|\Psi(t=0)\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Solve

Schrödinger’s equation given the initial conditions, and determine the ket of the kettle at later times.

Schrödinger’s equation says that

$$i\hbar \frac{d}{dt} \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = H \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = \hbar\omega \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = i\hbar\omega \begin{pmatrix} -\beta(t) \\ \alpha(t) \end{pmatrix},$$

Or,

$$\dot{\alpha}(t) = -\omega\beta(t), \quad \dot{\beta}(t) = \omega\alpha(t)$$

These equations aren’t that hard to solve by inspection, but one way to get it more directly is by taking another time derivative of the first equation, which yields

$$\ddot{\alpha}(t) = -\omega\dot{\beta}(t) = -\omega^2\alpha(t)$$

which has general solution

$$\alpha(t) = A\cos(\omega t) + B\sin(\omega t)$$

The boundary conditions that $\alpha(0) = 1$ and $\dot{\alpha}(0) = -\omega\beta(0) = 0$ tells us $A = 1$ and $B = 0$.

We then use $\dot{\alpha}(t) = -\omega\beta(t)$ to show that $\beta(t) = -\dot{\alpha}(t)/\omega = \sin(\omega t)$, and we have

$$|\Psi(t)\rangle = \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix}$$

- (b)[3] At time $t = \Delta t$, an observer checks whether the kettle has boiled yet. That is, he measures the quantum system using the boiled operator B , defined by

$B|0\rangle = 0|0\rangle$ and $B|1\rangle = 1|1\rangle$. What is the probability P_1 that the kettle has boiled at this time (i.e., that measuring B yields the eigenvalue 1)? If the kettle is boiled at this time, the total time $T = \Delta t$ is recorded.

To determine the probability that it has boiled, we calculate

$$P_1 = P(1) = \left| \langle 1 | \Psi(t) \rangle \right|^2 = \left| \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix} \right|^2 = \sin^2(\omega t)$$

For future reference, we will note also that the probability it has *not* boiled is $\cos^2(\omega t)$.

(c) [3] If the kettle is *not* boiled, what is the quantum state immediately after the measurement has been made?

According to postulate 5, the quantum state after a measurement of not boiled is

$$|\Psi(t^+)\rangle = \frac{1}{\sqrt{P(0)}} |0\rangle \langle 0 | \Psi(t) \rangle = \frac{1}{\sqrt{\cos^2(\omega t)}} |0\rangle \cos(\omega t) = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In other words, it is back in its original state. It might be noted that technically, the correct answer is $|0\rangle \text{sgn}(\cos(\omega t))$, where $\text{sgn}(\cos(\omega t)) = \pm 1$ depending on whether $\cos(\omega t)$ is positive or negative, but since this is only an overall phase, it doesn't make any difference any way.

(d) [3] After a second interval of Δt , the kettle is measured again to see if it has boiled. What is the probability P_2 that it is not boiled the first time, and it is boiled the second? If this occurs, the total time $T = 2\Delta t$ is recorded.

Since it ends up back in the original state, the probability after an equal interval must be exactly the same. The probability that it did not boil the first time, and it did boil the second, is

$$P_2 = (1 - P_1) P_1 = \cos^2(\omega t) \sin^2(\omega t).$$

(e) [5] The process is repeated until the kettle has actually boiled. What is the general formula P_n that it first boils on the n 'th measurement? Write a formula for the average time $\langle T \rangle = \langle n\Delta t \rangle$ that it takes for the kettle to boil. The formula below may be helpful.

The computation is similar. To first boil on the n 'th trial, we must have $n - 1$ failures, followed by success, so

$$P_n = (1 - P_1)^{n-1} P_1 = \cos^{2n-2}(\omega t) \sin^2(\omega t)$$

The average time is just the sum of the products of the probabilities and the time in each case, so

$$\begin{aligned}\langle T \rangle &= \sum_{n=1}^{\infty} P_n n \Delta t = \Delta t \sin^2(\omega \Delta t) \sum_{n=1}^{\infty} n \cos^{2n-2}(\omega t) \\ &= \Delta t \sin^2(\omega \Delta t) [1 + 2 \cos^2(\omega t) + 3 \cos^4(\omega t) + \dots]\end{aligned}$$

The formula inside the square brackets is identical with the formula below if we set $x = \cos^2(\omega \Delta t)$, so we have

$$\langle T \rangle = \frac{\Delta t \sin^2(\omega \Delta t)}{[1 - \cos^2(\omega \Delta t)]^2} = \frac{\Delta t \sin^2(\omega \Delta t)}{\sin^4(\omega \Delta t)} = \frac{\Delta t}{\sin^2(\omega \Delta t)}$$

(f) [2] Demonstrate that in the limit $\Delta t \rightarrow 0$, it takes forever for the kettle to boil.

$$\lim_{\Delta t \rightarrow 0} \langle T \rangle = \lim_{\Delta t \rightarrow 0} \left[\frac{\Delta t}{\sin^2(\omega \Delta t)} \right] = \lim_{\Delta t \rightarrow 0} \frac{\Delta t}{\omega^2 (\Delta t)^2} = \infty$$

(g) [3] Determine (numerically or otherwise) the optimal time Δt so that $\langle T \rangle = \langle n \Delta t \rangle$ will be minimized, so the kettle boils as quickly as possible.

The function will be minimized when the derivative vanishes, that is, when

$$0 = \frac{d \langle T \rangle}{dt} = \frac{\sin(\omega \Delta t) - 2(\omega \Delta t) \cos(\omega \Delta t)}{\sin^3(\omega \Delta t)} = \frac{1 - 2(\omega \Delta t) \cot(\omega \Delta t)}{\sin^2(\omega \Delta t)}$$

If we let $x = \omega \Delta t$, we are trying to find a root of $1 - 2x \cot(x)$. We can search for the solution of this equation with the help of a calculator, or with the help of Maple

> fsolve(1-2*x*cot(x), x, 0..2);

Maple tells us the solution is $x = 1.16556$, so

$$\Delta t_{\min} = 1.16556/\omega$$

Helpful formula: $1 + 2x + 3x^2 + 4x^3 + \dots = \frac{d}{dx} (1 + x + x^2 + x^3 + x^4 + \dots) = \frac{1}{(1-x)^2}$

2. [10] The commutation relations of the angular momentum operator L_z with the momentum operator \mathbf{P} were to be worked out in chapter 3, problem 3.
 (a) [3] Using these commutation relations, derive two non-trivial uncertainty relationships.

The commutation relations were

$$[L_z, P_x] = i\hbar P_y, \quad [L_z, P_y] = -i\hbar P_x, \quad [L_z, P_z] = 0$$

According to the generalized uncertainty relation, we therefore have

$$\Delta L_z \Delta P_x \geq \frac{1}{2} \hbar |\langle P_y \rangle|, \quad \Delta L_z \Delta P_y \geq \frac{1}{2} \hbar |\langle P_x \rangle|, \quad \Delta L_z \Delta P_z \geq 0$$

Since the left side of each of these expressions is the product of two positive numbers, the last inequality doesn't give us any information, but the other two inequalities suffice.

- (b) [3] Show that if you are in an eigenstate of any observable, the uncertainty in that observable is zero.

Let A be any observable, and $|a\rangle$ a normalized eigenstate with $A|a\rangle = a|a\rangle$.

Then

$$(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2 = \langle a | A^2 | a \rangle - \langle a | A | a \rangle^2 = a \langle a | A | a \rangle - (a \langle a | a \rangle)^2 = a^2 \langle a | a \rangle - a^2 = 0$$

- (c) [4] Show that if you are in an eigenstate of L_z , then you must have

$$\langle P_x \rangle = \langle P_y \rangle = 0.$$

According to our inequalities we found above, if we are in an eigenstate of L_z , then we have $\Delta L_z = 0$, and therefore

$$0 \geq \frac{1}{2} \hbar |\langle P_y \rangle|, \quad 0 \geq \frac{1}{2} \hbar |\langle P_x \rangle|.$$

Since absolute values are never negative, it follows that $|\langle P_y \rangle| = |\langle P_x \rangle| = 0$, which implies

$$\langle P_x \rangle = \langle P_y \rangle = 0.$$

3. [10] We will eventually discover that particles have spin, which is described by three operators $\mathbf{S} = (S_x, S_y, S_z)$ with commutation relations

$$[S_x, S_y] = i\hbar S_z, \quad [S_y, S_z] = i\hbar S_x, \quad [S_z, S_x] = i\hbar S_y$$

A particle in a magnetic field of magnitude B pointing in the z -direction will have Hamiltonian $H = -\mu B S_z$ where μ and B are constants.

- (a) [5] Derive formulas for the time derivative of all three components of $\langle \mathbf{S} \rangle$

We use the standard formula for the time evolution of any operator, namely

$$\begin{aligned} \frac{d}{dt} \langle S_x \rangle &= \frac{i}{\hbar} \langle [H, S_x] \rangle = -\frac{i\mu B}{\hbar} \langle [S_z, S_x] \rangle = -\frac{i\mu B}{\hbar} i\hbar \langle S_y \rangle = \mu B \langle S_y \rangle, \\ \frac{d}{dt} \langle S_y \rangle &= \frac{i}{\hbar} \langle [H, S_y] \rangle = -\frac{i\mu B}{\hbar} \langle [S_z, S_y] \rangle = -\frac{i\mu B}{\hbar} (-i\hbar) \langle S_x \rangle = -\mu B \langle S_x \rangle, \\ \frac{d}{dt} \langle S_z \rangle &= \frac{i}{\hbar} \langle [H, S_z] \rangle = -\frac{i\mu B}{\hbar} \langle [S_z, S_z] \rangle = -\frac{i\mu B}{\hbar} 0 = 0. \end{aligned}$$

- (b) [5] At time $t = 0$, the expectation values of $\langle \mathbf{S} \rangle$ are given by

$$\langle S_x \rangle_{t=0} = a, \quad \langle S_y \rangle_{t=0} = 0, \quad \langle S_z \rangle_{t=0} = b$$

Determine the expectation value $\langle \mathbf{S} \rangle_t$ at later times.

Since the time derivative of $\langle S_z \rangle$ vanishes, this will just remain constant. The expectation value of the other two operators, however, are related by

$$d \langle S_x \rangle / dt = \mu B \langle S_y \rangle, \quad d \langle S_y \rangle / dt = -\mu B \langle S_x \rangle.$$

One way to proceed is to take another derivative of the first equation, which yields

$$d^2 \langle S_x \rangle / dt^2 = \mu B d \langle S_y \rangle / dt = -\mu^2 B^2 \langle S_x \rangle.$$

This suggests solutions along the lines of

$$\langle S_x \rangle = \alpha \cos(\mu B t) + \beta \sin(\mu B t)$$

Taking the derivative we can find

$$\langle S_y \rangle = -\alpha \sin(\mu B t) + \beta \cos(\mu B t)$$

Our boundary conditions at $t = 0$ then tell us that $\alpha = a$ and $\beta = 0$. In summary, we have

$$\langle S_x \rangle_t = a \cos(\mu B t), \quad \langle S_y \rangle_t = -a \sin(\mu B t), \quad \langle S_z \rangle_t = b$$

4. [15] A particle of mass m lies in a one-dimensional infinite square well in the region $[0,a]$. At $t = 0$, the wave function in the allowed region is

$$\langle x | \Psi(t=0) \rangle = \sqrt{30/a^5} (ax - x^2).$$

(a) [8] Write the wave function in the form $|\Psi(t=0)\rangle = \sum_n c_n |\phi_n\rangle$

where $|\phi_n\rangle$ are the energy eigenstates. Check that the wave function is properly normalized, both in the original coordinate basis, and in the new basis, either analytically or numerically.

The integral we need is

$$c_n = \langle \psi_n | \Psi(t=0) \rangle = \int_0^a \psi_n^*(x) \Psi(x, t=0) dx = \sqrt{\frac{2}{a}} \sqrt{\frac{30}{a^5}} \int_0^a (ax - x^2) \sin\left(\frac{\pi nx}{a}\right) dx.$$

This isn't too hard to do ourselves, but let's let Maple do the work.

```
> assume(n::integer);integrate((a-x)*x*sin(Pi*n*x/a),x=0..a)
/a^3;
```

$$c_n = \frac{2\sqrt{60}}{\pi^3 n^3} [1 - (-1)^n] = \begin{cases} 8\sqrt{15}/\pi^3 n^3 & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

So our initial state is

$$|\Psi(0)\rangle = \sum_n c_n |\phi_n\rangle = \sum_{n \text{ odd}} \frac{8\sqrt{15}}{\pi^3 n^3} |\phi_n\rangle.$$

In the original basis, normalization is given by

$$\begin{aligned} \langle \Psi(0) | \Psi(0) \rangle &= \int_0^a |\langle x | \Psi(0) \rangle|^2 dx = \frac{30}{a^5} \int_0^a (ax - x^2)^2 dx = \frac{30}{a^5} \int_0^a (a^2 x^2 - 2ax^3 + x^4) dx \\ &= \frac{30}{a^5} \left(\frac{1}{3} a^2 x^3 - \frac{1}{2} ax^4 + \frac{1}{5} x^5 \right) \Big|_0^a = 1. \end{aligned}$$

In the new basis, the normalization condition is

$$\langle \Psi(0) | \Psi(0) \rangle = \sum_{n=0}^{\infty} |\langle \phi_n | \Psi(0) \rangle|^2 = \sum_{n=0}^{\infty} |c_n|^2 = \frac{960}{\pi^6} \sum_{n \text{ odd}} \frac{1}{n^6}.$$

The final sum can be done numerically, or you can do it exactly with Maple.

```
> sum(1/(2*n+1)^6,n=0..infinity);
```

$$\langle \Psi(0) | \Psi(0) \rangle = \frac{960}{\pi^6} \frac{\pi^6}{960} = 1.$$

(b) [5] Find the expectation value of the Hamiltonian H for this wave function at $t = 0$, both in the original coordinate basis and the eigenstate basis. Check that they are the same, either analytically or numerically.

In the original basis, we simply use the Hamiltonian, but in the allowed region the potential vanishes, so

$$\begin{aligned}\langle H \rangle &= \frac{1}{2m} \langle P^2 \rangle = -\frac{\hbar^2}{2m} \frac{30}{a^5} \int_0^a (ax - x^2) \frac{d^2}{dx^2} (ax - x^2) dx = \frac{15\hbar^2}{ma^5} 2 \int_0^a (ax - x^2) dx \\ &= \frac{30\hbar^2}{ma^5} \left(\frac{1}{2} ax^2 - \frac{1}{3} x^3 \right) \Big|_0^a = \frac{5\hbar^2}{ma^2}.\end{aligned}$$

In the final basis, we have

$$\begin{aligned}\langle H \rangle &= \langle \Psi(0) | H | \Psi(0) \rangle = \sum_n \langle \Psi(0) | H | \phi_n \rangle \langle \phi_n | \Psi(0) \rangle = \sum_n E_n \langle \Psi(0) | \phi_n \rangle \langle \phi_n | \Psi(0) \rangle \\ &= \sum_n |c_n|^2 E_n = \sum_{n \text{ odd}} \frac{960}{\pi^6 n^6} \frac{\pi^2 \hbar^2}{2ma^2} = \frac{480\hbar^2}{\pi^4 ma^2} \sum_{n \text{ odd}} \frac{1}{n^4}.\end{aligned}$$

Obviously, it worked. The final sum was done by Maple for us.

> `sum((2*n+1)^(-4), n=0..infinity);`

$$\langle H \rangle = \frac{480\hbar^2}{\pi^4 ma^2} \frac{\pi^4}{96} = \frac{5\hbar^2}{ma^2}.$$

Obviously, it worked.

(c) [2] Write the wave function $|\Psi(t)\rangle$ at all times.

At arbitrary time, then the wave function is

$$|\Psi(t)\rangle = \sum_n c_n |\phi_n\rangle \exp(-iE_n t/\hbar) = \sum_{n \text{ odd}} \frac{8\sqrt{15}}{\pi^3 n^3} |\phi_n\rangle \exp\left(-\frac{i\pi^2 \hbar n^2}{2ma^2} t\right).$$