Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 2

1. [10] A free particle of mass *m* in one dimension takes the form at *t* = 0

$$\Psi(x,t=0)=\psi(x)=(A/\pi)^{1/4}\exp\left(iKx-\tfrac{1}{2}Ax^2\right)$$

This is identical with chapter 1 problem 4. Find the wave at all subsequent times.

The procedure, as discussed in class, is to first find the Fourier transform, $\tilde{\psi}(k)$. This was found in problem 1.4, part a:

$$\tilde{\psi}(k)=(\pi A)^{-1/4}e^{-(k-K)^2/2A}\,.$$

Then the answer to the question is simply

$$\psi(x,t)=\int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}\tilde{\psi}(k)\exp\left(ikx-i\frac{\hbar k^2}{2m}t\right)=(\pi A)^{-1/4}\int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}\exp\left(ikx-i\frac{\hbar k^2}{2m}t-\frac{k^2-2kK+K^2}{2A}\right)$$

$$=(\pi A)^{-1/4}e^{-K^2/2A}\int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}\exp\left[-\left(\frac{i\hbar t}{2m}+\frac{1}{2A}\right)k^2+\left(ix+\frac{K}{A}\right)k\right]$$

$$=(\pi A)^{-1/4}\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi}{i\hbar t/2m+1/2A}}e^{-K^2/2A}\exp\left[\frac{(ix+K/A)^2}{4(i\hbar t/2m+1/2A)}\right]$$

$$=\left(\frac{A}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+iA\hbar t/m}}\exp\left[-\frac{K^2}{2A}+\frac{-Ax^2+2iKx+K^2/A}{2(1+iA\hbar t/m)}\right]$$

$$=\left(\frac{A}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+iA\hbar t/m}}\exp\left[\frac{-Ax^2+2iKx+K^2/A-(K^2/A)(1+iA\hbar t/m)}{2(1+iA\hbar t/m)}\right]$$

$$=\left(\frac{A}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+iA\hbar t/m}}\exp\left[\frac{-Ax^2+2iKx-i\hbar K^2t/m}{2(1+iA\hbar t/m)}\right].$$

It's messy, but it's finished, and there isn't much you can do to simplify it.

2. [10] One solution of the 2D Harmonic oscillator Schrodinger equation takes the form

$$\Psi(x,y,t) = (x+iy)e^{-A(x^2+y^2)/2}e^{-i\omega t}$$

(a) [3] Find the probability density $\rho(x,y,t)$ at all times.

$$\rho(x,y,t) = \Psi^*\Psi = (x-iy)e^{-A(x^2+y^2)/2}e^{i\omega t}(x+iy)e^{-A(x^2+y^2)/2}e^{-i\omega t} = (x^2+y^2)e^{-A(x^2+y^2)}.$$

(b) [4] Find the probability current $\mathbf{j}(x,y,t)$ at all times.

$$\mathbf{j} = \frac{\hbar}{m}\mathrm{Im}(\Psi^*\nabla\Psi) = \frac{\hbar}{m}\mathrm{Im}\left\{(x-iy)e^{-A(x^2+y^2)/2}e^{i\omega t}\left(\hat{\mathbf{x}}\frac{\partial}{\partial x}+\hat{\mathbf{y}}\frac{\partial}{\partial y}\right)\left[(x+iy)e^{-A(x^2+y^2)/2}e^{-i\omega t}\right]\right\}$$

$$= \frac{\hbar}{m}\mathrm{Im}\left[(x-iy)e^{-A(x^2+y^2)/2}\left\{\hat{\mathbf{x}}\left[1-Ax(x+iy)\right]+\hat{\mathbf{y}}\left[i-Ay(x+iy)\right]\right\}e^{-A(x^2+y^2)/2}\right]$$

$$= \frac{\hbar}{m}e^{-A(x^2+y^2)}\mathrm{Im}\left\{\hat{\mathbf{x}}\left[x-iy-Ax(x^2+y^2)\right]+\hat{\mathbf{y}}\left[ix+y-Ay(x^2+y^2)\right]\right\} = \frac{\hbar}{m}(-y\hat{\mathbf{x}}+x\hat{\mathbf{y}})e^{-A(x^2+y^2)}.$$

(c) [3] Check the local version of conservation of probability, i.e., show that your solution satisfies $\partial\rho/\partial t + \nabla\cdot\mathbf{j} = 0$

Since ρ is independent of time, the first term is zero.

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{j} = \frac{\partial j_x}{\partial x}+\frac{\partial j_y}{\partial y} = \frac{\hbar}{m}\left\{\frac{\partial}{\partial x}\left[-ye^{-A(x^2+y^2)}\right]+\frac{\partial}{\partial y}\left[xe^{-A(x^2+y^2)}\right]\right\}$$

$$= \frac{\hbar}{m}\left[2Axye^{-A(x^2+y^2)}-2Axye^{-A(x^2+y^2)}\right] = 0.$$

3. [10] A particle of mass *m* lies in the one-dimensional infinite square well, which has potential with allowed region $0 < x < a$. At $t = 0$, the wave function takes the form $\left(4/\sqrt{5a}\right)\sin^3(\pi x/a)$. Rewrite this in the form $\Psi(x, t=0) = \sum_i c_i \psi_i(x)$. Find the wave function $\Psi(x,t)$ at all later times. The identity $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$ may be helpful.

We will take advantage of the identity given, which we first rewrite as

$$\sin^3\theta = \tfrac{3}{4}\sin\theta - \tfrac{1}{4}\sin(3\theta)$$

So we have

$$\Psi(x, t=0) = \frac{4}{\sqrt{5a}}\left[\frac{3}{4}\sin\left(\frac{\pi x}{a}\right) - \frac{1}{4}\sin\left(\frac{3\pi x}{a}\right)\right] = \frac{1}{\sqrt{5a}}\left[3\sin\left(\frac{\pi x}{a}\right) - \sin\left(\frac{3\pi x}{a}\right)\right]$$
$$= \frac{1}{\sqrt{10}}\left[3\psi_1(x) - \psi_3(x)\right]$$

In other words, we have $c_1 = 3/\sqrt{10}$, $c_3 = -1/\sqrt{10}$, and the rest of the c_i's vanish.

We now substitute this into the general solution to yield

$$\Psi(x,t) = \sum_i c_i \psi_i(x) e^{-iE_i t/\hbar} = \frac{3}{\sqrt{10}}\psi_1(x) e^{-iE_1 t/\hbar} - \frac{1}{\sqrt{10}}\psi_3(x) e^{-iE_3 t/\hbar}$$
$$= \frac{1}{\sqrt{5a}}\left[3\sin\left(\frac{\pi x}{a}\right)\exp\left(-i\frac{\pi^2\hbar t}{2ma^2}\right) - \sin\left(\frac{3\pi x}{a}\right)\exp\left(-i\frac{9\pi^2\hbar t}{2ma^2}\right)\right].$$