

Physics 741 – Graduate Quantum Mechanics 1
Solutions to Chapter 2

1. [10] A free particle of mass m in one dimension takes the form at $t = 0$

$$\Psi(x, t = 0) = \psi(x) = (A/\pi)^{1/4} \exp\left(iKx - \frac{1}{2}Ax^2\right)$$

Note that this is identical with chapter 1 problem 4. Find the wave at all subsequent times.

The procedure, as discussed in class, is to first find the Fourier transform, $\tilde{\psi}(k)$. This was found in problem set C, problem 1a.

$$\tilde{\psi}(k) = (\pi A)^{-1/4} \exp\left[-(k - K)^2 / 2A\right]$$

Then the answer to the question is simply

$$\begin{aligned} \psi(x, t) &= \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \tilde{\psi}(k) \exp\left(ikx - i\frac{\hbar k^2}{2m}t\right) \\ &= (\pi A)^{-1/4} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \exp\left[ikx - i\frac{\hbar k^2}{2m}t - \frac{1}{2A}k^2 + \frac{1}{A}kK - \frac{1}{2A}K^2\right] \\ &= (\pi A)^{-1/4} \exp\left(-\frac{K^2}{2A}\right) \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \exp\left[-\left(\frac{i\hbar t}{2m} + \frac{1}{2A}\right)k^2 + \left(ix + \frac{K}{A}\right)k\right] \\ &= (\pi A)^{-1/4} \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{i\hbar t/2m + 1/2A}} \exp\left(-\frac{K^2}{2A}\right) \exp\left[\frac{(ix + K/A)^2}{4(i\hbar t/2m + 1/2A)}\right] \\ &= \left(\frac{A}{\pi}\right)^{1/4} \frac{1}{\sqrt{1 + iA\hbar t/m}} \exp\left(-\frac{K^2}{2A} + \frac{-Ax^2 + 2iKx + K^2/A}{2(1 + iA\hbar t/m)}\right) \\ &= \left(\frac{A}{\pi}\right)^{1/4} \frac{1}{\sqrt{1 + iA\hbar t/m}} \exp\left(\frac{-Ax^2 + 2iKx + K^2/A - (K^2/A)(1 + iA\hbar t/m)}{2(1 + iA\hbar t/m)}\right) \\ &= \left(\frac{A}{\pi}\right)^{1/4} \frac{1}{\sqrt{1 + iA\hbar t/m}} \exp\left(\frac{-Ax^2 + 2iKx - i\hbar K^2 t/m}{2(1 + iA\hbar t/m)}\right) \end{aligned}$$

It's messy, but it's finished, and there isn't much you can do to simplify it.

2. [10] One solution of the 2D Harmonic oscillator Schrodinger equation looks like this:

$$\Psi(x, y, t) = (x + iy) e^{-A(x^2+y^2)/2} e^{-i\omega t}$$

- (a) [3] Find the probability density $\rho(x, y, t)$ at all times.

$$\rho(x, y, t) = \Psi^* \Psi = (x - iy) e^{-A(x^2+y^2)/2} e^{i\omega t} (x + iy) e^{-A(x^2+y^2)/2} e^{-i\omega t} = (x^2 + y^2) e^{-A(x^2+y^2)}.$$

- (b) [4] Find the probability current $\mathbf{j}(x, y, t)$ at all times.

$$\begin{aligned} \mathbf{j} &= \frac{\hbar}{m} \text{Im}(\Psi^* \nabla \Psi) \\ &= \frac{\hbar}{m} \text{Im} \left\{ (x - iy) e^{-A(x^2+y^2)/2} e^{i\omega t} \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} \right) \left[(x + iy) e^{-A(x^2+y^2)/2} e^{-i\omega t} \right] \right\} \\ &= \frac{\hbar}{m} \text{Im} \left\{ (x - iy) e^{-A(x^2+y^2)/2} \left(\hat{\mathbf{x}} [1 - Ax(x + iy)] + \hat{\mathbf{y}} [i - Ay(x + iy)] \right) e^{-A(x^2+y^2)/2} \right\} \\ &= \frac{\hbar}{m} e^{-A(x^2+y^2)} \text{Im} \left\{ \hat{\mathbf{x}} [x - iy - Ax(x^2 + y^2)] + \hat{\mathbf{y}} [ix + y - Ay(x^2 + y^2)] \right\} \\ &= \frac{\hbar}{m} (-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}) e^{-A(x^2+y^2)}. \end{aligned}$$

- (c) [3] Check the local version of conservation of probability, *i.e.*, show that your solution satisfies $\partial\rho/\partial t + \nabla \cdot \mathbf{j} = 0$

Since ρ is independent of time, the first term is zero.

$$\begin{aligned} \frac{\partial\rho}{\partial t} + \nabla \cdot \mathbf{j} &= \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} = \frac{\hbar}{m} \left\{ \frac{\partial}{\partial x} \left[-ye^{-A(x^2+y^2)} \right] + \frac{\partial}{\partial y} \left[xe^{-A(x^2+y^2)} \right] \right\} \\ &= \frac{\hbar}{m} \left[2Axye^{-A(x^2+y^2)} - 2Axye^{-A(x^2+y^2)} \right] = 0 \end{aligned}$$

3. [10] A particle of mass m lies in the one-dimensional infinite square well, which has potential with allowed region $0 < x < a$. At $t = 0$, the wave function takes the form $(4/\sqrt{5a})\sin^3(\pi x/a)$. Rewrite this in the form $\Psi(x, t=0) = \sum_i c_i \psi_i(x)$.

Find the wave function $\Psi(x, t)$ at all later times. You may find the identity $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$ useful.

We will take advantage of the identity given, which we first rewrite as

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta)$$

So we have

$$\begin{aligned}\Psi(x, t=0) &= \frac{4}{\sqrt{5a}} \left[\frac{3}{4} \sin\left(\frac{\pi x}{a}\right) - \frac{1}{4} \sin\left(\frac{3\pi x}{a}\right) \right] = \frac{3}{\sqrt{5a}} \sin\left(\frac{\pi x}{a}\right) - \frac{1}{\sqrt{5a}} \sin\left(\frac{3\pi x}{a}\right) \\ &= \frac{3}{\sqrt{10}} \psi_1(x) - \frac{1}{\sqrt{10}} \psi_3(x)\end{aligned}$$

In other words, we have $c_1 = 3/\sqrt{10}$, $c_3 = -1/\sqrt{10}$, and the rest of the c_i 's vanish.

The general solution is

$$\Psi(x, t) = \sum_i c_i \psi_i(x) e^{-iE_i t/\hbar}$$

In this case, we have

$$\Psi(x, t) = \frac{3}{\sqrt{5a}} \sin\left(\frac{\pi x}{a}\right) \exp\left(-i \frac{\pi^2 \hbar t}{2ma^2}\right) - \frac{1}{\sqrt{5a}} \sin\left(\frac{3\pi x}{a}\right) \exp\left(-i \frac{9\pi^2 \hbar t}{2ma^2}\right)$$

4. [15] A particle of mass m and energy E scatters from the negative- x direction off of a delta-function potential: $V(x) = \lambda\delta(x)$.

(a) [4] For the regions $x < 0$ and $x > 0$, find general equations for the wave, eliminating any terms that are physically inappropriate.

In both regions, the potential vanishes, and therefore the solutions just look like $e^{\pm ikx}$. On the right side, however, we want only a transmitted wave, so we throw out one of the solutions and write

$$\begin{aligned}\psi_I(x) &= Ae^{ikx} + Be^{-ikx}, \\ \psi_{II}(x) &= Ce^{ikx}, \\ E &= \hbar^2 k^2 / 2m.\end{aligned}$$

(b) [5] Integrate Schrödinger's time-independent equation across the boundary to obtain an equation relating the derivative of the wave function on either side of the boundary. Will the wave function itself be continuous?

As in class, we start with Schrödinger's time independent equation

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x) + \lambda \delta(x) \psi(x) = E \psi(x)$$

and integrate it across the boundary at $x = 0$:

$$\int_{-\varepsilon}^{\varepsilon} \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x) + \lambda \delta(x) \psi(x) \right\} dx = E \int_{-\varepsilon}^{\varepsilon} \psi(x) dx,$$

$$-\frac{\hbar^2}{2m} \frac{\partial}{\partial x} \psi(x) \Big|_{-\varepsilon}^{\varepsilon} + \lambda \psi(0) = 0,$$

$$-\frac{\hbar^2}{2m} \psi'_{II}(0) + \frac{\hbar^2}{2m} \psi'_{I}(0) + \lambda \psi(0) = 0.$$

This implies a finite discontinuity in the derivative, which means that the function itself is presumably continuous, so we also have $\psi_{II}(0) = \psi_I(0)$.

(c) [6] Solve the equations and deduce the transmission and reflection coefficients T and R . Check that $T + R = 1$.

The continuity of the wave function tells us that

$$A + B = C,$$

while the first derivative condition tells us that

$$-\frac{\hbar^2}{2m} ikC + \frac{\hbar^2}{2m} ik(A - B) + \lambda(A + B) = 0.$$

If we substitute the first equation into the second, this becomes

$$\frac{\hbar^2}{2m} ik(A - B) + \lambda(A + B) = \frac{\hbar^2}{2m} ik(A + B), \quad \text{so that } \lambda(A + B) = \hbar^2 ikB/m,$$

We can then solve this and find

$$\frac{B}{A} = \frac{\lambda m}{ik\hbar^2 - \lambda m} \quad \text{and} \quad \frac{C}{A} = 1 + \frac{B}{A} = \frac{ik\hbar^2}{ik\hbar^2 - \lambda m}$$

The reflection and transmission coefficients are then

$$R = \frac{|j_B|}{j_A} = \frac{|B|^2 k}{|A|^2 k} = \frac{\lambda^2 m^2}{k^2 \hbar^4 + \lambda^2 m^2} \quad \text{and} \quad T = \frac{j_C}{j_A} = \frac{|C|^2 k}{|A|^2 k} = \frac{k^2 \hbar^4}{k^2 \hbar^4 + \lambda^2 m^2}$$

In this form it is obvious that $R + T = 1$.