

Physics 741 – Graduate Quantum Mechanics 1
Solutions to Chapter 2

1. [10] A free particle of mass m in one dimension takes the form at $t = 0$

$$\Psi(x, t = 0) = \psi(x) = (A/\pi)^{1/4} \exp\left(iKx - \frac{1}{2}Ax^2\right)$$

This is identical with chapter 1 problem 4. Find the wave at all subsequent times.

The procedure, as discussed in class, is to first find the Fourier transform, $\tilde{\psi}(k)$. This was found in problem 1.4, part a:

$$\tilde{\psi}(k) = (\pi A)^{-1/4} \exp\left[-\frac{(k-K)^2}{2A}\right].$$

Then the answer to the question is simply

$$\begin{aligned} \psi(x, t) &= \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \tilde{\psi}(k) \exp\left(ikx - i\frac{\hbar k^2}{2m}t\right) \\ &= (\pi A)^{-1/4} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \exp\left(ikx - i\frac{\hbar k^2}{2m}t - \frac{k^2 - 2kK + K^2}{2A}\right) \\ &= (\pi A)^{-1/4} \exp\left(-\frac{K^2}{2A}\right) \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \exp\left[-\left(\frac{i\hbar t}{2m} + \frac{1}{2A}\right)k^2 + \left(ix + \frac{K}{A}\right)k\right] \\ &= (\pi A)^{-1/4} \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{i\hbar t/2m + 1/2A}} \exp\left(-\frac{K^2}{2A}\right) \exp\left[\frac{(ix + K/A)^2}{4(i\hbar t/2m + 1/2A)}\right] \\ &= \left(\frac{A}{\pi}\right)^{1/4} \frac{1}{\sqrt{1 + iA\hbar t/m}} \exp\left[-\frac{K^2}{2A} + \frac{-Ax^2 + 2iKx + K^2/A}{2(1 + iA\hbar t/m)}\right] \\ &= \left(\frac{A}{\pi}\right)^{1/4} \frac{1}{\sqrt{1 + iA\hbar t/m}} \exp\left[\frac{-Ax^2 + 2iKx + K^2/A - (K^2/A)(1 + iA\hbar t/m)}{2(1 + iA\hbar t/m)}\right] \\ &= \left(\frac{A}{\pi}\right)^{1/4} \frac{1}{\sqrt{1 + iA\hbar t/m}} \exp\left[\frac{-Ax^2 + 2iKx - i\hbar K^2 t/m}{2(1 + iA\hbar t/m)}\right]. \end{aligned}$$

It's messy, but it's finished, and there isn't much you can do to simplify it.

2. [10] One solution of the 2D Harmonic oscillator Schrodinger equation takes the form

$$\Psi(x, y, t) = (x + iy) e^{-A(x^2+y^2)/2} e^{-i\omega t}$$

- (a) [3] Find the probability density $\rho(x, y, t)$ at all times.

$$\rho(x, y, t) = \Psi^* \Psi = (x - iy) e^{-A(x^2+y^2)/2} e^{i\omega t} (x + iy) e^{-A(x^2+y^2)/2} e^{-i\omega t} = (x^2 + y^2) e^{-A(x^2+y^2)}.$$

- (b) [4] Find the probability current $\mathbf{j}(x, y, t)$ at all times.

$$\begin{aligned} \mathbf{j} &= \frac{\hbar}{m} \text{Im}(\Psi^* \nabla \Psi) \\ &= \frac{\hbar}{m} \text{Im} \left\{ (x - iy) e^{-A(x^2+y^2)/2} e^{i\omega t} \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} \right) \left[(x + iy) e^{-A(x^2+y^2)/2} e^{-i\omega t} \right] \right\} \\ &= \frac{\hbar}{m} \text{Im} \left[(x - iy) e^{-A(x^2+y^2)/2} \left\{ \hat{\mathbf{x}} [1 - Ax(x + iy)] + \hat{\mathbf{y}} [i - Ay(x + iy)] \right\} e^{-A(x^2+y^2)/2} \right] \\ &= \frac{\hbar}{m} e^{-A(x^2+y^2)} \text{Im} \left\{ \hat{\mathbf{x}} [x - iy - Ax(x^2 + y^2)] + \hat{\mathbf{y}} [ix + y - Ay(x^2 + y^2)] \right\} \\ &= \frac{\hbar}{m} (-y \hat{\mathbf{x}} + x \hat{\mathbf{y}}) e^{-A(x^2+y^2)}. \end{aligned}$$

- (c) [3] Check the local version of conservation of probability, i.e., show that your solution satisfies $\partial \rho / \partial t + \nabla \cdot \mathbf{j} = 0$

Since ρ is independent of time, the first term is zero.

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} &= \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} = \frac{\hbar}{m} \left\{ \frac{\partial}{\partial x} \left[-y e^{-A(x^2+y^2)} \right] + \frac{\partial}{\partial y} \left[x e^{-A(x^2+y^2)} \right] \right\} \\ &= \frac{\hbar}{m} \left[2Axy e^{-A(x^2+y^2)} - 2Axy e^{-A(x^2+y^2)} \right] = 0. \end{aligned}$$

3. [10] A particle of mass m lies in the one-dimensional infinite square well, which has potential with allowed region $0 < x < a$. At $t = 0$, the wave function takes the form $(4/\sqrt{5a})\sin^3(\pi x/a)$. Rewrite this in the form $\Psi(x, t=0) = \sum_i c_i \psi_i(x)$.

Find the wave function $\Psi(x, t)$ at all later times. The identity $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$ may be helpful.

We will take advantage of the identity given, which we first rewrite as

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta)$$

So we have

$$\begin{aligned}\Psi(x, t=0) &= \frac{4}{\sqrt{5a}} \left[\frac{3}{4} \sin\left(\frac{\pi x}{a}\right) - \frac{1}{4} \sin\left(\frac{3\pi x}{a}\right) \right] = \frac{3}{\sqrt{5a}} \sin\left(\frac{\pi x}{a}\right) - \frac{1}{\sqrt{5a}} \sin\left(\frac{3\pi x}{a}\right) \\ &= \frac{3}{\sqrt{10}} \psi_1(x) - \frac{1}{\sqrt{10}} \psi_3(x)\end{aligned}$$

In other words, we have $c_1 = 3/\sqrt{10}$, $c_3 = -1/\sqrt{10}$, and the rest of the c_i 's vanish.

The general solution is

$$\Psi(x, t) = \sum_i c_i \psi_i(x) e^{-iE_i t/\hbar}.$$

In this case, we have

$$\Psi(x, t) = \frac{3}{\sqrt{5a}} \sin\left(\frac{\pi x}{a}\right) \exp\left(-i \frac{\pi^2 \hbar t}{2ma^2}\right) - \frac{1}{\sqrt{5a}} \sin\left(\frac{3\pi x}{a}\right) \exp\left(-i \frac{9\pi^2 \hbar t}{2ma^2}\right)$$

4. [20] A particle of mass m and energy E scatters from the negative- x direction off of a delta-function potential: $V(x) = \lambda\delta(x)$.

(a) [4] For the regions $x < 0$ and $x > 0$, find general equations for the wave, eliminating any terms that are physically inappropriate.

In both regions, the potential vanishes, and therefore the solutions just look like $e^{\pm ikx}$. On the right side, however, we want only a transmitted wave, so we throw out one of the solutions and write

$$\begin{aligned}\psi_I(x) &= Ae^{ikx} + Be^{-ikx}, \\ \psi_{II}(x) &= Ce^{ikx}, \\ E &= \frac{\hbar^2 k^2}{2m}.\end{aligned}$$

(b) [5] Integrate Schrödinger's time-independent equation across the boundary to obtain an equation relating the derivative of the wave function on either side of the boundary. Will the wave function itself be continuous?

As in class, we start with Schrödinger's time independent equation

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x) + \lambda \delta(x) \psi(x) = E \psi(x)$$

and integrate it across the boundary at $x = 0$:

$$\int_{-\varepsilon}^{\varepsilon} \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x) + \lambda \delta(x) \psi(x) \right] dx = E \int_{-\varepsilon}^{\varepsilon} \psi(x) dx,$$

$$-\frac{\hbar^2}{2m} \frac{\partial}{\partial x} \psi(x) \Big|_{-\varepsilon}^{\varepsilon} + \lambda \psi(0) = 0,$$

$$-\frac{\hbar^2}{2m} \psi'_II(0) + \frac{\hbar^2}{2m} \psi'_I(0) + \lambda \psi(0) = 0.$$

This implies a finite discontinuity in the derivative, which means that the function itself is presumably continuous, so we also have $\psi_{II}(0) = \psi_I(0)$.

(c) [6] Solve the equations and deduce the transmission and reflection coefficients T and R . Check that $T + R = 1$.

The continuity of the wave function tells us that

$$A + B = C,$$

while the first derivative condition tells us that

$$-\frac{\hbar^2}{2m} ikC + \frac{\hbar^2}{2m} ik(A - B) + \lambda(A + B) = 0.$$

If we substitute the first equation into the second, this becomes

$$\frac{\hbar^2}{2m} ik(A - B) + \lambda(A + B) = \frac{\hbar^2}{2m} ik(A + B), \quad \text{so that } \lambda(A + B) = \hbar^2 ikB/m,$$

We can then solve this and find

$$\frac{B}{A} = \frac{\lambda m}{ik\hbar^2 - \lambda m} \quad \text{and} \quad \frac{C}{A} = 1 + \frac{B}{A} = \frac{ik\hbar^2}{ik\hbar^2 - \lambda m}.$$

The reflection and transmission coefficients are then

$$R = \frac{|j_B|}{j_A} = \frac{|B|^2 k}{|A|^2 k} = \frac{\lambda^2 m^2}{k^2 \hbar^4 + \lambda^2 m^2} \quad \text{and} \quad T = \frac{j_C}{j_A} = \frac{|C|^2 k}{|A|^2 k} = \frac{k^2 \hbar^4}{k^2 \hbar^4 + \lambda^2 m^2}$$

In this form it is obvious that $R + T = 1$.

(d) [5] A delta function potential $V(x) = \lambda\delta(x)$ can be thought of as a very narrow potential of height V_0 and width d , with $V_0d = \lambda$. Show that you can get the same result using (2.38), where you can approximate $\sinh(\alpha d) = \alpha d$ since d is small.

This is really pretty straightforward. We simply use the approximation and the suggested formula, and we find

$$\begin{aligned} T &= \frac{4E(V_0 - E)}{4E(V_0 - E) + V_0^2 \sinh^2(\alpha d)} = \frac{4E(V_0 - E)}{4E(V_0 - E) + V_0^2 \alpha^2 d^2} \\ &= \frac{4E(V_0 - E)}{4E(V_0 - E) + V_0^2 d^2 2m(V_0 - E)/\hbar^2} = \frac{2(\hbar^2 k^2/2m)}{2(\hbar^2 k^2/2m) + (V_0 d)^2 m/\hbar^2} = \frac{\hbar^4 k^2}{\hbar^4 k^2 + \lambda^2 m^2}. \end{aligned}$$

Indeed, it came out exactly as before.

6. [10] A particle of mass m is trapped in a potential $V(x) = -\lambda\delta(x)$. Show that there is only one bound state and find its energy.

Let the energy be E , which must be negative. Schrödinger's equation then gives

$$\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = -E\psi(x) - \lambda\delta(x)\psi(x),$$

Which, away from $x = 0$, has solutions $\psi(x) = e^{\pm\beta x}$, where $E = -\hbar^2\beta^2/2m$. We divide it into two regions, region I with $x < 0$ and region II with $x > 0$. To keep the wave function from diverging, we must pick

$$\psi_I(x) = Ae^{\beta x} \quad \text{and} \quad \psi_{II}(x) = Be^{-\beta x}.$$

If we integrate Schrödinger's equation across the boundary at $x = 0$, we find,

$$\begin{aligned} \frac{\hbar^2}{2m} \int_{-\varepsilon}^{\varepsilon} \frac{d^2\psi}{dx^2} dx &= -E \int_{-\varepsilon}^{\varepsilon} \psi(x) dx - \lambda \int_{-\varepsilon}^{\varepsilon} \delta(x)\psi(x) dx, \\ \frac{\hbar^2}{2m} [\psi'_{II}(0) - \psi'_I(0)] &= -\lambda\psi(0). \end{aligned}$$

Since the derivative of the wave function has a finite discontinuity, the wave function must be continuous at the boundary, so we have $\psi_I(0) = \psi_{II}(0)$, which implies $A = B$.

Then the equation just derived for the discontinuity of the derivative become

$$\frac{\hbar^2}{2m} [-A\beta - A\beta] = -\lambda A, \quad \text{or} \quad \beta = \frac{m\lambda}{\hbar^2}.$$

Substituting this in for the equation for the energy, we find $E = -\hbar^2\beta^2/2m = -m\lambda^2/2\hbar^2$.