Physics 742 – Graduate Quantum Mechanics 2

Solutions to Chapter 15

7. [15] A spin ½ particle of mass *m* lies in a one-dimensional spin-dependant potential $H_0 = P^2/2m + \frac{1}{2}m\omega^2 X^2 \left|+\right\rangle\left\langle+\right|$. The potential only affects particles in a spin-up state.

(a) [4] Find the discrete energy eigenstates for spin-up ($\left|+,i\right\rangle$) and the continuum energy eigenstates for spin-down ($\left|-,\beta\right\rangle$). Also, identify their energies.

For spin up particles, the Hamiltonian is $H_0 = P^2/2m + \frac{1}{2}m\omega^2 X^2$, which is a harmonic oscillator potential. The eigenstates are labeled $\left|+,n\right\rangle$, where the + denotes spin up, and have energy $\hbar\omega\left(n+\frac{1}{2}\right)$. The ground state, for example, is given by

$$\psi_{0,+}(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right)\left|+\right\rangle .$$

For spin down particles, the Hamiltonian is $H_0 = P^2/2m$, which is just the free particle Hamiltonian. These states can be chosen to be labeled by their wave number, which we could denote by $\left|-,k\right\rangle$. They have energy $\hbar^2 k^2/2m$. Properly normalized, they look like

$$\psi_{k,-}(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}\left|-\right\rangle .$$

(b) [11] At *t* = 0, a spin-dependant perturbation of the form $V = \hbar\gamma\sigma_x$, where σ_x is a Pauli matrix, is turned on. Calculate the rate Γ at which the spin-up ground state "decays" to a continuum state.

To leading order, the rate for the transition should be given by

$$\Gamma = 2\pi\hbar^{-1}\left|\left\langle F\right|V\left|I\right\rangle\right|^2 \delta\left(E_f - E_i\right) = 2\pi\hbar\gamma^2\left|\left\langle -,k\right|\sigma_x\left|+,0\right\rangle\right|^2 \delta\left(E_f - E_i\right)$$

$$= 2\pi\hbar\gamma^2\left|\sqrt[4]{\frac{m\omega}{\pi\hbar}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx \exp\left(-\frac{m\omega x^2}{2\hbar} - ikx\right)\left\langle -\right|\sigma_x\left|+\right\rangle\right|^2 \delta\left(E_f - E_i\right)$$

$$= \hbar\gamma^2\sqrt{\frac{m\omega}{\pi\hbar}}\left|\sqrt{\frac{2\pi\hbar}{m\omega}}\exp\left(-\frac{\hbar k^2}{2m\omega}\right)\right|^2 \delta\left(E_f - E_i\right) = 2\hbar\gamma^2\sqrt{\frac{\pi\hbar}{m\omega}}\exp\left(-\frac{\hbar k^2}{m\omega}\right)\delta\left(\frac{\hbar^2 k^2}{2m} - \frac{\hbar\omega}{2}\right).$$

This would be the rate to a particular wave number *k* if the states were properly normalized. To finish the problem, we simply sum (integrate) over the continuum of final states to yield

$$\Gamma = \int_{-\infty}^{\infty} 2\hbar\gamma^2\sqrt{\frac{\pi\hbar}{m\omega}}\exp\left(-\frac{\hbar k^2}{m\omega}\right)\delta\left(\frac{\hbar^2 k^2}{2m} - \frac{\hbar\omega}{2}\right)dk = 2\hbar\gamma^2\sqrt{\frac{\pi\hbar}{m\omega}}\sum_k \exp\left(-\frac{\hbar k^2}{m\omega}\right)\left|\frac{\hbar^2 k}{m}\right|^{-1} .$$

The sum is over all values of k where the delta function vanishes. It is pretty easy to see that these points are $k = \pm\sqrt{\omega m/\hbar}$. There are two such points, and each of them contributes exactly the same amount, so our rate is

$$\Gamma = 2 \cdot 2\hbar\gamma^2 \sqrt{\frac{\pi\hbar}{m\omega}} \exp\left(-\frac{\hbar}{m\omega}\frac{m\omega}{\hbar}\right)\frac{m}{\hbar^2}\sqrt{\frac{\hbar}{m\omega}} = \frac{4\sqrt{\pi}\gamma^2}{e\omega}.$$

A rather bizarre looking answer, but neat.