Physics 742 – Graduate Quantum Mechanics 2

Solutions to Chapter 15

1. [10] An isolated tritium (hydrogen) atom ^3H has its electron in the ground state when it suddenly radioactively decays to ^3He, (helium) but the nucleus stays in the same place (no recoil). What is the probability that the atom remains in the ground state? What is the probability that it goes into each of the *n* = 2 states $|2lm\rangle$?

The probability is just $P(I \to F) = |\langle nlm|100\rangle|^2$, but the eigenstate of the initial Hamiltonian are *not* the same as the eigenstate of the final Hamiltonian. The angular integrals will vanish unless $l = m = 0$, and in this case the angular integral yields one, so we have

$$\langle nlm|100\rangle = \delta_{l0}\delta_{m0}\int_0^\infty r^2 R'_{n0}(r) R_{10}(r)\,dr$$

The prime doesn't denote derivative, but rather that the radial wave function must be evaluated for Helium, which has the same wave function except $a_0 \to \frac{1}{2}a_0$. We'll let Maple finish it for us, using the online wave functions for hydrogen.

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> for n to 6 do integrate(r^2*subs(a=a/2,radial(n,0))
  *radial(1,0),r=0..infinity)^2 end do;
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Just for fun, I worked out the first six solutions, with the probabilities listed below. The other $|2lm\rangle$ states, of course, have probability zero.

$$P(1s \to 1s) = \tfrac{512}{729}, \quad P(1s \to 3s) = \tfrac{124416}{9765625}, \quad P(1s \to 5s) = \tfrac{1166400000}{678223072849},$$
$$P(1s \to 2s) = \tfrac{1}{4}, \quad P(1s \to 4s) = \tfrac{2048}{531441}, \quad P(1s \to 6s) = \tfrac{243}{262144}.$$

If you add up all these probabilities, it comes to about 97.2%. Probably most of the remaining 2.8% represents the probability that the electron becomes unbound.

2. **[20] A neutral boron atom has a total angular momentum $l = 1$ and spin $s = ½$. In the absence of a magnetic field, the lowest energy states might be listed as $\left|l, s, j, m_j\right\rangle = \left|1, \tfrac{1}{2}, j, m_j\right\rangle$, with the $j = \tfrac{3}{2}$ state having higher energy. The atom is placed in a region of space where a magnetic field is being turned on in the +z direction. At first, the spin-orbit coupling dominates, but at late times the magnetic interactions dominate.**

   (a) **[3] Which of the nine operators L, S and J will commute with the Hamiltonian at all times? Note that the state must remain an eigenstate of this operator at all times.**

The presence of the magnetic field in the z –direction does not destroy rotational invariance around the z-axis. Since this is generated by J_z, J_z will commute with the Hamiltonian. None of the others will. Hence the J_z eigenvalue is always good.

   (b) **[7] At strong magnetic fields, the states are dominated by the magnetic field. The eigenstates are approximately $\left|l, s, m_l, m_s\right\rangle = \left|1, \tfrac{1}{2}, m_l, m_s\right\rangle$. For each possible value of $m_j = m_l + m_s$, deduce which state has the lower energy. Atoms in strong magnetic fields are discussed in chapter 9, section E.**

The energy of the state $\left|l, s, m_l, m_s\right\rangle$ has a magnetic contribution

$$E_{\text{mag}} = \frac{eB\hbar}{2\mu}\left(m_l + gm_s\right).$$

Although this was computed specifically for hydrogen, it is not hard to see that it applies in general. Now, for any given value of m_j, we have $m_j = m_l + m_s$, so that we can rewrite this expression as

$$E_{\text{mag}} = \frac{eB\hbar}{2\mu}\left[m_j + \left(g-1\right)m_s\right].$$

Since $g > 1$ (it's around 2), we conclude that for fixed m_j, the one with higher m_s value will have higher energy, in other words, the state $\left|l, s, m_l, m_s\right\rangle = \left|1, \tfrac{1}{2}, m_j - \tfrac{1}{2}, \tfrac{1}{2}\right\rangle$ has more energy than $\left|l, s, m_l, m_s\right\rangle = \left|1, \tfrac{1}{2}, m_j + \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle$. Of course, if $m_j = \pm\tfrac{3}{2}$, only one of these states is allowed.

(c) [10] If we start with a particular value of $\left|l,s,j,m_j\right\rangle$ (six cases), calculate which states $\left|l,s,m_l,m_s\right\rangle$ it might evolve into, assuming the magnetic field increases (i) adiabatically (slowly) or (ii) suddenly. When relevant, give the corresponding probabilities. The relevant Clebsch-Gordan coefficients are given in eq. (8.18).

In the adiabatic case, the higher energy state will always evolve into the higher energy state, and the lower into the lower. For each possible value of m_j, we simply map the higher energy state to the higher, and lower to lower.

$$\begin{aligned}
m_j &= +\tfrac{3}{2}: \quad \left|1,\tfrac12,\tfrac32,+\tfrac32\right\rangle \to \left|1,\tfrac12,+1,+\tfrac12\right\rangle, \\
m_j &= +\tfrac{1}{2}: \quad \left|1,\tfrac12,\tfrac32,+\tfrac12\right\rangle \to \left|1,\tfrac12,0,+\tfrac12\right\rangle \quad \text{and} \quad \left|1,\tfrac12,\tfrac12,+\tfrac12\right\rangle \to \left|1,\tfrac12,+1,-\tfrac12\right\rangle, \\
m_j &= -\tfrac{1}{2}: \quad \left|1,\tfrac12,\tfrac32,-\tfrac12\right\rangle \to \left|1,\tfrac12,-1,+\tfrac12\right\rangle \quad \text{and} \quad \left|1,\tfrac12,\tfrac12,-\tfrac12\right\rangle \to \left|1,\tfrac12,0,-\tfrac12\right\rangle, \\
m_j &= -\tfrac{3}{2}: \quad \left|1,\tfrac12,\tfrac32,-\tfrac32\right\rangle \to \left|1,\tfrac12,-1,-\tfrac12\right\rangle.
\end{aligned}$$

The probabilities in every case are 1. In the sudden approximation, on the other hand, there will be probabilities, since any of the six states might evolve into other states with the same m_j values. Everything turns into Clebsch-Gordan coefficients. We have

$$\begin{aligned}
P\left(\left|1,\tfrac12,\tfrac32,\tfrac32\right\rangle \to \left|1,\tfrac12,1,\tfrac12\right\rangle\right) &= \left|\left\langle 1,\tfrac12,1,\tfrac12 \middle| \tfrac32,\tfrac32\right\rangle\right|^2 = \left|\sqrt{\left(1+\tfrac12+\tfrac32\right)/3}\right|^2 = 1, \\
P\left(\left|1,\tfrac12,\tfrac32,\tfrac12\right\rangle \to \left|1,\tfrac12,0,\tfrac12\right\rangle\right) &= \left|\left\langle 1,\tfrac12,0,\tfrac12 \middle| \tfrac32,\tfrac12\right\rangle\right|^2 = \left|\sqrt{\left(1+\tfrac12+\tfrac12\right)/3}\right|^2 = \tfrac23, \\
P\left(\left|1,\tfrac12,\tfrac32,\tfrac12\right\rangle \to \left|1,\tfrac12,+1,-\tfrac12\right\rangle\right) &= \left|\left\langle 1,\tfrac12,1,-\tfrac12 \middle| \tfrac32,\tfrac12\right\rangle\right|^2 = \left|\sqrt{\left(1+\tfrac12-\tfrac12\right)/3}\right|^2 = \tfrac13, \\
P\left(\left|1,\tfrac12,\tfrac12,\tfrac12\right\rangle \to \left|1,\tfrac12,0,+\tfrac12\right\rangle\right) &= \left|\left\langle 1,\tfrac12,0,\tfrac12 \middle| \tfrac12,\tfrac12\right\rangle\right|^2 = \left|-\sqrt{\left(1+\tfrac12-\tfrac12\right)/3}\right|^2 = \tfrac13, \\
P\left(\left|1,\tfrac12,\tfrac12,\tfrac12\right\rangle \to \left|1,\tfrac12,1,-\tfrac12\right\rangle\right) &= \left|\left\langle 1,\tfrac12,1,-\tfrac12 \middle| \tfrac12,\tfrac12\right\rangle\right|^2 = \left|\sqrt{\left(1+\tfrac12+\tfrac12\right)/3}\right|^2 = \tfrac23, \\
P\left(\left|1,\tfrac12,\tfrac32,-\tfrac12\right\rangle \to \left|1,\tfrac12,-1,\tfrac12\right\rangle\right) &= \left|\left\langle 1,\tfrac12,-1,\tfrac12 \middle| \tfrac32,-\tfrac12\right\rangle\right|^2 = \left|\sqrt{\left(1+\tfrac12+\left(-\tfrac12\right)\right)/3}\right|^2 = \tfrac13, \\
P\left(\left|1,\tfrac12,\tfrac32,-\tfrac12\right\rangle \to \left|1,\tfrac12,0,-\tfrac12\right\rangle\right) &= \left|\left\langle 1,\tfrac12,0,-\tfrac12 \middle| \tfrac32,-\tfrac12\right\rangle\right|^2 = \left|\sqrt{\left(1+\tfrac12-\left(-\tfrac12\right)\right)/3}\right|^2 = \tfrac23, \\
P\left(\left|1,\tfrac12,\tfrac12,-\tfrac12\right\rangle \to \left|1,\tfrac12,-1,\tfrac12\right\rangle\right) &= \left|\left\langle 1,\tfrac12,-1,\tfrac12 \middle| \tfrac12,-\tfrac12\right\rangle\right|^2 = \left|-\sqrt{\left(1+\tfrac12-\left(-\tfrac12\right)\right)/3}\right|^2 = \tfrac23, \\
P\left(\left|1,\tfrac12,\tfrac12,-\tfrac12\right\rangle \to \left|1,\tfrac12,0,-\tfrac12\right\rangle\right) &= \left|\left\langle 1,\tfrac12,0,-\tfrac12 \middle| \tfrac12,-\tfrac12\right\rangle\right|^2 = \left|\sqrt{\left(1+\tfrac12+\left(-\tfrac12\right)\right)/3}\right|^2 = \tfrac13, \\
P\left(\left|1,\tfrac12,\tfrac32,-\tfrac32\right\rangle \to \left|1,\tfrac12,-1,-\tfrac12\right\rangle\right) &= \left|\left\langle 1,\tfrac12,-1,-\tfrac12 \middle| \tfrac32,-\tfrac32\right\rangle\right|^2 = \left|\sqrt{\left(1+\tfrac12-\left(-\tfrac32\right)\right)/3}\right|^2 = 1.
\end{aligned}$$

That was more than a little scary. Note in every case that the probabilities for all the final states given an initial state add up to one.