Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 14

4. [10] In class we found the scattering cross section for Coulomb scattering by a charge *q′* from a point charge *q* located at the origin, with potential $V(\mathbf{r}) = k_e qq'/r$. We needed the Fourier transform, which turned out to be $\int d^3\mathbf{r}V(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}} = 4\pi k_e qq'/\mathbf{K}^2$.

(a) [3] Place the source *q* not at the origin, but at $\mathbf{r} = \mathbf{a}$. What is the potential now? What is the Fourier transform now? Hint: **Don't actually do the work, just shift your integration variable, and use previous work. Convince yourself (and me) that the differential cross-section is unchanged.**

We need to replace *r* in the denominator of the potential with $|\mathbf{r} - \mathbf{a}|$. We then simply shift our integration variable to $\mathbf{r} \to \mathbf{r} + \mathbf{a}$. We find

$$\int d^3\mathbf{r}V(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}} = k_e qq'\int \frac{d^3\mathbf{r}}{|\mathbf{r} - \mathbf{a}|}e^{-i\mathbf{K}\cdot\mathbf{r}} = k_e qq'\int \frac{d^3\mathbf{r}}{|\mathbf{r} + \mathbf{a} - \mathbf{a}|}e^{-i\mathbf{K}\cdot(\mathbf{r}+\mathbf{a})} = \frac{4\pi k_e qq'}{\mathbf{K}^2}e^{-i\mathbf{K}\cdot\mathbf{a}}$$

The only change is a phase, but since you end up squaring the amplitude, it doesn't change things at all.

(b) [4] Suppose we replaced *q* with a series of charges q_i located at several locations $\mathbf{a}_i$. What would be the Fourier transform now? What if, instead of a series of discrete charges q_i, we had a charge distribution $\rho(\mathbf{r})$ spread around in space?

The potential from a series of charges is obviously

$$V(\mathbf{r}) = \sum_i \frac{k_e q_i q'}{|\mathbf{r} - \mathbf{a}_i|}.$$

The Fourier transform of this, as a consequence, is clearly just the sum from each of the separate charges

$$\int d^3\mathbf{r}V(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}} = \frac{4\pi k_e q'}{\mathbf{K}^2}\sum_i q_i e^{-i\mathbf{K}\cdot\mathbf{a}_i} .$$

For a charge distribution, the obvious generalization is

$$\int d^3\mathbf{r}V(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}} = \frac{4\pi k_e q'}{\mathbf{K}^2}\int d^3\mathbf{r}\,\rho(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}} .$$

(c) [3] Show that the differential cross-section for a charge q' scattering off of a charge distribution $\rho(\mathbf{r})$ is given by

$$\frac{d\sigma}{d\Omega}=\frac{4\mu^2k_e^2q'^2}{\hbar^4\left(\mathbf{K}^2\right)^2}\left|\int d^3\mathbf{r}\rho(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}}\right|^2$$

where $\mathbf{K}=\mathbf{k}'-\mathbf{k}$, the change in the wave number.

We simply substitute our previous result into the formula for the cross-section, which yields

$$\frac{d\sigma}{d\Omega}=\frac{\mu^2}{4\pi^2\hbar^4}\left|\frac{4\pi k_e q'}{\mathbf{K}^2}\int d^3\mathbf{r}\,\rho(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}}\right|^2=\frac{4\mu^2k_e^2q'^2}{\hbar^4\left(\mathbf{K}^2\right)^2}\left|\int d^3\mathbf{r}\,\rho(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}}\right|^2.$$

5. [15] An electron of mass *m* scatters from a neutral hydrogen atom in the ground state located at the origin.

(a) [7] What is the charge distribution for a neutral hydrogen atom? Don't forget the nucleus! What is the Fourier transform of the charge?

The nucleus has charge e and is locate at the origin, so we can model it with a charge distribution $\rho(\mathbf{r})=e\delta^3(\mathbf{r})$. The electron is spread out in a wave function given by $\rho(\mathbf{r})=-e\left|\psi(\mathbf{r})\right|^2$. Using the explicit form of the wave function for the ground state, we therefore have

$$\rho(\mathbf{r})=e\delta^3(\mathbf{r})-\frac{e}{\pi a_0^3}e^{-2r/a_0}.$$

We will have to keep our e's straight. When computing the Fourier transform, the charge distribution is spherically symmetric, so there's no harm in assuming $\mathbf{K}$ is in the z-direction. The Fourier transform of this is

$$\begin{aligned}\int d^3\mathbf{r}\rho(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}}&=e-\frac{e}{\pi a_0^3}\int d^3\mathbf{r}\,e^{-2r/a_0}e^{-i\mathbf{K}\cdot\mathbf{r}}=e-\frac{2e}{a_0^3}\int_{-1}^{1}d\cos\theta\int_0^\infty r^2dr\exp\left(-2r/a_0-iKr\cos\theta\right)\\&=e-\frac{4e}{a_0^3}\int_{-1}^{1}\left(2/a_0+iK\cos\theta\right)^{-3}d\cos\theta=e+\frac{2e}{iKa_0^3}\left(2/a_0+iK\cos\theta\right)^{-2}\Big|_{\cos\theta=-1}^{\cos\theta=1}\\&=e+\frac{2e}{iKa_0^3}\left[\frac{1}{\left(2/a_0+iK\right)^2}-\frac{1}{\left(2/a_0-iK\right)^2}\right]=e+\frac{2e}{iKa_0^3}\frac{-8iK/a_0}{\left(4/a_0^2+K^2\right)^2}\\&=e-\frac{16e}{\left(4+K^2a_0^2\right)^2}=e\frac{8K^2a_0^2+K^4a_0^4}{\left(4+K^2a_0^2\right)^2}.\end{aligned}$$

Obviously, this vanishes if $K=0$.

(b) [8] Find the differential and total cross section in this case.

We simply use the formula from problem 4. We find

$$\frac{d\sigma}{d\Omega}=\frac{4m^2k_e^2q'^2}{\hbar^4\left(\mathbf{K}^2\right)^2}\left|\int d^3\mathbf{r}\,\rho\left(\mathbf{r}\right)e^{-i\mathbf{K}\cdot\mathbf{r}}\right|^2=\frac{4m^2k_e^2e^4\left(8a_0^2+K^2a_0^4\right)^2}{\hbar^4\left(4+K^2a_0^2\right)^4}=a_0^2\frac{\left[4+k^2a_0^2\left(1-\cos\theta\right)\right]^2}{\left[2+k^2a_0^2\left(1-\cos\theta\right)\right]^4},$$

where we have used the identity $\mathbf{K}^2=2k^2\left(1-\cos\theta\right)$ to simplify our expression, as well as $a_0=\hbar^2/mk_ee^2$. We now have a single nasty integral to do, which is why God invented Maple.

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> assume(A>0);integrate((4+A-A*x)^2/(2+A-A*x)^4,x=-1..1);
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$$\sigma=2\pi a_0^2\int_{-1}^{1}\frac{\left[4+k^2a_0^2\left(1-\cos\theta\right)\right]^2}{\left[2+k^2a_0^2\left(1-\cos\theta\right)\right]^4}d\cos\theta$$
$$=4\pi a_0^2\frac{1+\frac{3}{2}k^2a_0^2+\frac{7}{12}k^4a_0^4}{\left(1+k^2a_0^2\right)^3}$$

Note that the final factor goes to one in the low energy limit, so we have a finite cross section $4\pi a_0^2$, which then decreases as we increase our energy.