Physics 742 – Graduate Quantum Mechanics 1

Solutions to Chapter 13

4. [10] We would like to know the hyperfine splitting for ordinary hydrogen ^{1}H and deuterium ^{2}H.

(a) For ^{1}H, the electron orbits a spin- ½ ($i=\frac{1}{2}$) proton with g_p = 5.5857. Find the frequency of radio waves emitted from hydrogen, and compare with the experimental value of 1420.4 MHz.

The energy shift is given by

$$\begin{aligned}\varepsilon' &= \frac{\mu_0 g_p g e^2}{8\pi m m_p}\frac{4\pi}{3}\left\langle 1,0,\tfrac{1}{2},f,m_f\right|\delta^3(\mathbf{R})\mathbf{I}\cdot\mathbf{S}\left|1,0,\tfrac{1}{2},f,m_f\right\rangle \\ &= \frac{\mu_0 g_p g e^2\hbar^2}{12 m m_p}\left(f^2+f-i^2-i-\tfrac{3}{4}\right)\left|\psi_{100}(0)\right|^2\end{aligned}$$

We are really interested in the differences in energy between the two possible states, which have $f=i\pm\frac{1}{2}$, so we have

$$\Delta\left(f^2+f-i^2-i-\tfrac{3}{4}\right)=\left(i+\tfrac{1}{2}\right)^2+\left(i+\tfrac{1}{2}\right)-\left(i-\tfrac{1}{2}\right)^2-\left(i-\tfrac{1}{2}\right)=2i+1$$

Substitute this into the previous equation, along with $\mu_0 c^2=4\pi k_e$ and the explicit form of the wave function to get

$$\begin{aligned}\Delta E &= \frac{4\pi k_e g_p g e^2\hbar^2}{12 m m_p c^2 \pi a_0^3}(2i+1)=\frac{g_p g\left(k_e e^2\right)^4 m^3}{3 m_p m c^2\hbar^4}(2i+1)=\tfrac{1}{3}g g_p(2i+1)\left(\frac{m}{m_p}\right)\alpha^4\left(mc^2\right) \\ &= g_P(2i+1)\frac{2.0023\left(5.11\times10^5\text{ eV}\right)}{3(1836.153)(137.0360)^4}=g_P(2i+1)5.267\times10^{-7}\text{eV}\end{aligned}$$

The corresponding frequency is then

$$\begin{aligned}f &= \frac{\Delta E}{h}=\frac{0.5267\times10^{-6}g_P(2i+1)\text{eV}}{4.136\times10^{-15}\text{ eV}\cdot\text{s}}=127.35 g_p(2i+1)\text{ MHz} \\ &= 127.35(5.5857)(2)\text{ MHz}=1422.6\text{ MHz}.\end{aligned}$$

This answer is about 0.15% off. The main source of this discrepancy is that, when calculating the wave function at the origin, we used the electron mass when we should have used the reduced electron mass. If you correct for this, the answer is less than 0.01% off, and I don't know why.

(b) For ^{2}H, the electron orbits a spin-1 ($i = 1$) deuteron (bound state of proton and neutron), with magnetic moment given by $\boldsymbol{\mu} = \left(g_d e/2m_p\right)\mathbf{I}$. *If* the proton and neutron have no orbital angular momentum around each other, then we would predict that it all comes from the spin, and then $g_d = \tfrac{1}{2}\left(g_p + g_n\right)$, (because half of its spin comes from each), where $g_n = -3.8261$. Find a formula for the splitting of the energy in this case (you will have to redo some of the computations), and then calculate the hyperfine frequency for deuterium, comparing to the experimental value of 327.4 MHz.

Because of the way we did the computation, we need do very little new computation. We find $g_d = \tfrac{1}{2}\left(g_p + g_n\right) = \tfrac{1}{2}\left(5.5857 - 3.8261\right) = 0.8798$, and for the deuteron, $2i + 1 = 3$, so

$$f = 127.35\left(0.8798\right)\left(3\right)\text{MHz} = 336.1\ \text{MHz}.$$

The error is 2.67% now, probably mostly due to the approximation that there is no orbital angular momentum for the nucleons.

5. [10] In section D, we worked out the Zeeman (weak magnetic field) effect only in the case where the total spin of the atom is 0. Work out the splitting for all values of *j* and m_j in the case of spin-1/2. Formulas for the relevant Clebsch-Gordan coefficients can be found in eq. (8.18).

We start with equation (14.39):

$$\Delta E_B\left(m_j\right)=\frac{eB\hbar}{2m}\sum_{m_l,m_s}\left(m_l+gm_s\right)\left|\left\langle ls;m_lm_s\mid jm_j\right\rangle\right|^2$$

Now, we are working with $s=\frac{1}{2}$, which means j can take on any value from $\left|l-\frac{1}{2}\right|$ to $l+\frac{1}{2}$, *i.e.* $j=l\pm\frac{1}{2}$. Hence there will be two cases for any fixed l. There are only two terms in the sum, because m_s can only take on the values $\pm\frac{1}{2}$, and therefore $m_l=m_j\mp\frac{1}{2}$. So

$$\begin{aligned}
\Delta E_B\left(m_j,j=l-\tfrac{1}{2}\right)&=\frac{eB\hbar}{2m}\left\{\begin{array}{l}\left(m_j-\frac{1}{2}+g\frac{1}{2}\right)\left|\left\langle l,\frac{1}{2};m_j-\frac{1}{2},\frac{1}{2}\middle|l-\frac{1}{2},m_j\right\rangle\right|^2\\+\left(m_j+\frac{1}{2}-g\frac{1}{2}\right)\left|\left\langle l,\frac{1}{2};m_j+\frac{1}{2},-\frac{1}{2}\middle|l-\frac{1}{2},m_j\right\rangle\right|^2\end{array}\right\}\\
&=\frac{eB\hbar}{2m}\left\{\left(m_j-\tfrac{1}{2}+g\tfrac{1}{2}\right)\frac{l+\frac{1}{2}-m_j}{2l+1}+\left(m_j+\tfrac{1}{2}-g\tfrac{1}{2}\right)\frac{l+\frac{1}{2}+m_j}{2l+1}\right\}\\
&=\frac{eB\hbar}{2m}\left[m_j-\left(g-1\right)\frac{m_j}{2l+1}\right]=\frac{eB\hbar}{2m}\left(\frac{2l+2-g}{2l+1}\right)m_j,\\
\Delta E_B\left(m_j,j=l+\tfrac{1}{2}\right)&=\frac{eB\hbar}{2m}\left\{\begin{array}{l}\left(m_j-\frac{1}{2}+g\frac{1}{2}\right)\left|\left\langle l,\frac{1}{2};m_j-\frac{1}{2},\frac{1}{2}\middle|l+\frac{1}{2},m_j\right\rangle\right|^2\\+\left(m_j+\frac{1}{2}-g\frac{1}{2}\right)\left|\left\langle l,\frac{1}{2};m_j+\frac{1}{2},-\frac{1}{2}\middle|l+\frac{1}{2},m_j\right\rangle\right|^2\end{array}\right\}\\
&=\frac{eB\hbar}{2m}\left\{\left(m_j-\tfrac{1}{2}+g\tfrac{1}{2}\right)\frac{l+\frac{1}{2}+m_j}{2l+1}+\left(m_j+\tfrac{1}{2}-g\tfrac{1}{2}\right)\frac{l+\frac{1}{2}-m_j}{2l+1}\right\}\\
&=\frac{eB\hbar}{2m}\left[m_j+\left(g-1\right)\frac{m_j}{2l+1}\right]=\frac{eB\hbar}{2m}\left(\frac{2l+g}{2l+1}\right)m_j.
\end{aligned}$$

In each case, we see that the energies are simply split by a factor proportional to m_j, so we end up with $2j+1$ completely separated spin states. Note that in the absence of the second term, the effect would be identical to that of a total spin zero state. The spin-orbit coupling has a tendency to enhance the effect for $j=l+\frac{1}{2}$ and suppress it for $j=l-\frac{1}{2}$. Note, however, that the case $j=l-\frac{1}{2}$ is not allowed if $l=0$, so only the second formula is allowed, where the expression in parentheses becomes g, the factor for a free electron.