Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 12

8. [20] A particle of mass *m* lies in a two dimensional harmonic oscillator plus a perturbation $H = P_x^2/2m + P_y^2/2m + \frac{1}{2}m\omega^2\left(X^2+Y^2\right) + \gamma X^3 P_y$, where γ is very small.

(a) [1] What are the eigenstates and eigenenergies of this in the limit $\gamma = 0$?

In this limit, we have simply the sum of two harmonic oscillators, with eigenstates $|ij\rangle$, where *i* and *j* are non-negative integers, and energy $\varepsilon_{ij} = \hbar\omega\left(i+j+1\right)$.

(b) [9] Find the ground state and energy to first order and second order in γ respectively.

The ground state $|00\rangle$ is non-degenerate. If we denote the annihilation operators in the *x*- and *y*-direction as a_x and a_y respectively, then we find

$$W|00\rangle = \gamma\left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} i\left(\frac{\hbar m\omega}{2}\right)^{\frac{1}{2}}\left(a_x + a_x^\dagger\right)^3\left(a_y^\dagger - a_y\right)|00\rangle = \frac{i\gamma\hbar^2}{4m\omega}\left(a_x + a_x^\dagger\right)^2|11\rangle$$

$$= \frac{i\gamma\hbar^2}{4m\omega}\left(a_x + a_x^\dagger\right)\left(|01\rangle + \sqrt{2}|21\rangle\right) = \frac{i\gamma\hbar^2}{4m\omega}\left(3|11\rangle + \sqrt{6}|31\rangle\right).$$

It is evident there will be no first-order contribution to the energy. To second order, the contribution will be

$$E_{00} = \varepsilon_{00} + \sum_{ij}\frac{\left|\langle ij|W|00\rangle\right|^2}{\varepsilon_{00}-\varepsilon_{ij}} = \hbar\omega + \frac{\left|\langle 11|W|00\rangle\right|^2}{-2\hbar\omega} + \frac{\left|\langle 31|W|00\rangle\right|^2}{-4\hbar\omega} = \hbar\omega - \left|\frac{i\gamma\hbar^2}{4m\omega}\right|^2\frac{1}{\hbar\omega}\left(\frac{3^2}{2}+\frac{6}{4}\right)$$

$$= \hbar\omega - \frac{3\gamma^2\hbar^3}{8m^2\omega^3}$$

The state vector is given by

$$|\psi_{00}\rangle = |00\rangle + \sum_{ij}|ij\rangle\frac{\langle ij|W|00\rangle}{\varepsilon_{00}-\varepsilon_{ij}} = \hbar\omega + |11\rangle\frac{\langle 11|W|00\rangle}{-2\hbar\omega} + |31\rangle\frac{\langle 31|W|00\rangle}{-4\hbar\omega}$$

$$= |00\rangle - \frac{i\gamma\hbar^2}{4m\omega}\frac{1}{\hbar\omega}\left(\frac{3}{2}|11\rangle + \frac{\sqrt{6}}{4}|31\rangle\right) = |00\rangle - \frac{i\gamma\hbar}{16m\omega^2}\left(6|11\rangle + \sqrt{6}|31\rangle\right).$$

(c) [10] Find the first excited states and energy to zeroth and first order in γ respectively.

The first excited states are $|01\rangle$ and $|10\rangle$, which are degenerate. As a consequence, we must use degenerate perturbation theory. We see that

$$W|10\rangle = \frac{i\gamma\hbar^2}{4m\omega}\left(a_x + a_x^\dagger\right)^3\left(a_y^\dagger - a_y\right)|10\rangle = \frac{i\gamma\hbar^2}{4m\omega}\left(a_x + a_x^\dagger\right)^3|11\rangle = \frac{i\gamma\hbar^2}{4m\omega}\left(a_x + a_x^\dagger\right)^2\left(|01\rangle + \sqrt{2}|21\rangle\right)$$
$$= \frac{i\gamma\hbar^2}{4m\omega}\left(a_x + a_x^\dagger\right)\left(3|11\rangle + \sqrt{6}|31\rangle\right) = \frac{i\gamma\hbar^2}{4m\omega}\left(3|01\rangle + 6\sqrt{2}|21\rangle + \sqrt{24}|41\rangle\right),$$
$$W|01\rangle = \frac{i\gamma\hbar^2}{4m\omega}\left(a_x + a_x^\dagger\right)^3\left(a_y^\dagger - a_y\right)|01\rangle = \frac{i\gamma\hbar^2}{4m\omega}\left(a_x + a_x^\dagger\right)^2\left(a_y^\dagger - a_y\right)|11\rangle$$
$$= \frac{i\gamma\hbar^2}{4m\omega}\left(a_x + a_x^\dagger\right)\left(a_y^\dagger - a_y\right)\left(|01\rangle + \sqrt{2}|21\rangle\right) = \frac{i\gamma\hbar^2}{4m\omega}\left(a_y^\dagger - a_y\right)\left(3|11\rangle + \sqrt{6}|31\rangle\right)$$
$$= \frac{i\gamma\hbar^2}{4m\omega}\left(3\sqrt{2}|12\rangle + 2\sqrt{3}|32\rangle - 3|10\rangle - \sqrt{6}|30\rangle\right)$$

We now can quickly see that the $\tilde{W}$ matrix is

$$\tilde{W} = \begin{pmatrix} \langle 10|W|10\rangle & \langle 10|W|01\rangle \\ \langle 01|W|10\rangle & \langle 01|W|01\rangle \end{pmatrix} = \frac{3\gamma\hbar^2}{4m\omega}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The last matrix is well-known; it is σ_y, one of the Pauli matrices. Its eigenvalues are ± 1, and its normalized eigenvectors are

$$|\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

This tells us which combinations to take. If we define the states

$$|\pm\rangle = \tfrac{1}{\sqrt{2}}\left(|10\rangle \pm i|01\rangle\right)$$

then these states will be the first excited eigenstates, to leading order, and will have energies

$$E_\pm = \varepsilon + w_\pm = 2\hbar\omega \pm \frac{3\gamma\hbar^2}{4m\omega}.$$