

Physics 741 – Graduate Quantum Mechanics 1
Solutions to Chapter 12

6. [20] A particle of mass m lies in one dimension with Hamiltonian near the origin of $H = P^2/2m + \frac{1}{2}m\omega^2 X^2 + \gamma X^3$ where γ is very small.

(a) [8] Find the energy of *all* quantum states to second order in γ .

If we treat γ as zero, then we have a harmonic oscillator, which has eigenstates $|n\rangle$ and energies $\hbar\omega(n + \frac{1}{2})$. Letting our perturbation act on an arbitrary state, we have

$$\begin{aligned} W|n\rangle &= \gamma \left(\frac{\hbar}{2m\omega} \right)^{3/2} (a + a^\dagger)^3 |n\rangle = \gamma \left(\frac{\hbar}{2m\omega} \right)^{3/2} (a + a^\dagger)^2 (\sqrt{n+1}|n+1\rangle + \sqrt{n}|n-1\rangle) \\ &= \gamma \left(\frac{\hbar}{2m\omega} \right)^{3/2} (a + a^\dagger) (\sqrt{(n+1)(n+2)}|n+2\rangle + (2n+1)|n\rangle + \sqrt{n(n-1)}|n-2\rangle) \\ &= \gamma \left(\frac{\hbar}{2m\omega} \right)^{3/2} \left(\sqrt{(n+1)(n+2)(n+3)}|n+3\rangle + (3n+3)\sqrt{n+1}|n+1\rangle \right. \\ &\quad \left. + 3n\sqrt{n}|n-1\rangle + \sqrt{n(n-1)(n-2)}|n-3\rangle \right) \end{aligned}$$

The energies of our states, therefore, are

$$\begin{aligned} E_n &= \varepsilon_n + \langle n|W|n\rangle + \sum_{m \neq n} \frac{|\langle m|W|n\rangle|^2}{\varepsilon_n - \varepsilon_m} \\ &= \hbar\omega(n + \frac{1}{2}) + 0 + \frac{|\langle n-3|W|n\rangle|^2}{3\hbar\omega} + \frac{|\langle n-1|W|n\rangle|^2}{\hbar\omega} + \frac{|\langle n+1|W|n\rangle|^2}{-\hbar\omega} + \frac{|\langle n+3|W|n\rangle|^2}{-3\hbar\omega} \\ &= \hbar\omega(n + \frac{1}{2}) + \gamma^2 \left(\frac{\hbar}{2m\omega} \right)^3 \frac{1}{\hbar\omega} \left[\frac{1}{3}n(n-1)(n-2) + (3n)^2 n - (3n+3)^2 (n+1) \right] \\ &= \hbar\omega(n + \frac{1}{2}) + \frac{\gamma^2 \hbar^2}{8m^3 \omega^4} \left(\frac{1}{3}n^3 - n^2 + \frac{2}{3}n + 9n^3 - 9n^3 - 27n^2 - 27n - 9 - \frac{1}{3}n^3 - 2n^2 - \frac{11}{3}n - 2 \right) \\ &= \hbar\omega(n + \frac{1}{2}) - \frac{\gamma^2 \hbar^2}{8m^3 \omega^4} (30n^2 + 30n + 11) \end{aligned}$$

(b) [5] Find the eigenstates of H to first order in γ .

The states are given, to first order in γ , by

$$\begin{aligned}
|\psi_n\rangle &= |n\rangle + \sum_{m \neq n} \frac{|m\rangle \langle n|W|p\rangle}{\varepsilon_n - \varepsilon_m} \\
&= |n\rangle + |n+3\rangle \frac{\langle n+3|W|n\rangle}{-3\hbar\omega} + |n+1\rangle \frac{\langle n+1|W|n\rangle}{-\hbar\omega} + |n-1\rangle \frac{\langle n-1|W|n\rangle}{\hbar\omega} + |n-3\rangle \frac{\langle n-3|W|n\rangle}{3\hbar\omega} \\
&= |n\rangle + \frac{\gamma}{3\hbar\omega} \left(\frac{\hbar}{2m\omega} \right)^{3/2} \left(-\sqrt{(n+3)(n+2)(n+1)} |n+3\rangle - 9(n+1)^{3/2} |n+1\rangle + 9n^{3/2} |n-1\rangle \right. \\
&\quad \left. + \sqrt{n(n-1)(n-2)} |n-3\rangle \right)
\end{aligned}$$

(c) [7] Find the expectation value $\langle \psi_n | X | \psi_n \rangle$ for the eigenstates $|\psi_n\rangle$ to first order in γ

We simply take

$$\begin{aligned}
\langle \psi_n | X | \psi_n \rangle &= \sqrt{\frac{\hbar}{2m\omega}} \left[\langle n | + \sum_{m \neq n} \frac{\langle n|W|m\rangle \langle m|}{\varepsilon_n - \varepsilon_m} \right] (a + a^\dagger) \left[|n\rangle + \sum_{m \neq n} \frac{|m\rangle \langle m|W|n\rangle}{\varepsilon_n - \varepsilon_m} \right] \\
&= \sqrt{\frac{\hbar}{2m\omega}} \sum_{m \neq n} \frac{1}{\varepsilon_n - \varepsilon_m} \left[\langle n|W|m\rangle \langle m|(a + a^\dagger)|n\rangle + \langle n|(a + a^\dagger)|m\rangle \langle m|W|n\rangle \right] \\
&= \sqrt{\frac{\hbar}{2m\omega}} \left[\frac{\sqrt{n}}{\hbar\omega} (\langle n|W|n-1\rangle + \langle n-1|W|n\rangle) + \frac{\sqrt{n+1}}{-\hbar\omega} (\langle n|W|n+1\rangle + \langle n+1|W|n\rangle) \right] \\
&= \frac{2\gamma}{\hbar\omega} \sqrt{\frac{\hbar}{2m\omega}} \left(\frac{\hbar}{2m\omega} \right)^{3/2} \left[\sqrt{n} 3n\sqrt{n} - \sqrt{n+1} (3n+3)\sqrt{n+1} \right] \\
&= \frac{6\gamma}{\hbar\omega} \left(\frac{\hbar}{2m\omega} \right)^2 \left[n^2 - (n+1)^2 \right] = -\frac{3\gamma\hbar(2n+1)}{2m^2\omega^3}
\end{aligned}$$

7. [15] A particle of mass m lies in a one-dimensional infinite slightly tilted square well

$$V(x) = \begin{cases} bx & \text{if } 0 < x < a \\ \infty & \text{otherwise} \end{cases}$$

where b is small.

(a) [2] What are the normalized eigenstates and energies if $b = 0$?

For $b = 0$, we just have an infinite square well, with wave functions and energies

$$\begin{aligned}
\phi_n(x) &= \sqrt{\frac{2}{a}} \sin\left(\frac{\pi nx}{a}\right) \\
\varepsilon_n &= \frac{\pi^2 \hbar^2 n^2}{2ma^2}
\end{aligned}$$

(b) [6] Find the eigenstates and energies to first order in b .

The energies are given, to first order in b , by

$$E_n = \varepsilon_n + \langle \phi_n | W | \phi_n \rangle = \frac{\pi^2 \hbar^2 n^2}{2ma^2} + b \frac{2}{a} \int_0^a x \sin^2 \left(\frac{\pi nx}{a} \right) dx = \frac{\pi^2 \hbar^2 n^2}{2ma^2} + b \frac{2}{a} \frac{a^2}{4} = \frac{\pi^2 \hbar^2 n^2}{2ma^2} + \frac{ba}{2}$$

To find the wave functions, we need to find

$$\langle \phi_p | W | \phi_n \rangle = b \frac{2}{a} \int_0^a x \sin \left(\frac{\pi nx}{a} \right) \sin \left(\frac{\pi px}{a} \right) dx = \frac{4banp \left[(-1)^{n+p} - 1 \right]}{\pi^2 (n^2 - p^2)^2}$$

We are avoiding the letter m so as not to confuse it with the mass. The expression in []'s vanishes when $n + p$ is even, and is -2 when $n + p$ is odd. So we have

$$\begin{aligned} |\psi_n\rangle &= |\phi_n\rangle + \sum_{p \neq n} |\phi_p\rangle \frac{\langle \phi_p | W | \phi_n \rangle}{\varepsilon_n - \varepsilon_p} = |\phi_n\rangle + \sum_{\substack{p+n \\ \text{odd}}} \frac{-8banp}{\pi^2 (n^2 - p^2)^2} \frac{2ma^2}{\pi^2 \hbar^2 (n^2 - p^2)} |\phi_p\rangle \\ &= |\phi_n\rangle - \frac{16bma^3}{\pi^4 \hbar^2} \sum_{\substack{p+n \\ \text{odd}}} \frac{np}{(n^2 - p^2)^3} |\phi_p\rangle. \end{aligned}$$

(c) [7] Find the ground state energy to second order in b . If you can't do the sums exactly, do them numerically.

Using our formula, we have

$$\begin{aligned} E_1 &= \varepsilon_1 + \langle \phi_1 | W | \phi_1 \rangle + \sum_{p \neq 1} \frac{|\langle \phi_p | W | \phi_1 \rangle|^2}{\varepsilon_1 - \varepsilon_p} = \frac{\pi^2 \hbar^2}{2ma^2} + \frac{ba}{2} + \sum_{p \text{ even}} \left[\frac{-8bap}{\pi^2 (1 - p^2)^2} \right]^2 \frac{2ma^2}{\pi^2 \hbar^2 (1 - p^2)} \\ &= \frac{\pi^2 \hbar^2}{2ma^2} + \frac{ba}{2} - \frac{128mb^2 a^4}{\pi^6 \hbar^2} \sum_{p \text{ even}} \frac{p^2}{(p^2 - 1)^5} \end{aligned}$$

The final sum can be performed numerically very quickly, since the terms fall as p^{-8} , or we can do it analytically by hand or with the help of Maple.

> `sum(4*n^2/(4*n^2-1)^5,n=1..infinity);`

$$\begin{aligned} E_1 &= \frac{\pi^2 \hbar^2}{2ma^2} + \frac{ba}{2} - \frac{128mb^2 a^4}{\pi^6 \hbar^2} \left(\frac{5\pi^2}{1024} - \frac{\pi^4}{3072} \right) = \frac{\pi^2 \hbar^2}{2ma^2} + \frac{ba}{2} - \frac{(15 - \pi^2) mb^2 a^4}{24\pi^4 \hbar^2} \\ &= \frac{\pi^2 \hbar^2}{2ma^2} + \frac{ba}{2} - 0.002195 \frac{mb^2 a^4}{\hbar^2} \end{aligned}$$