Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 11

4. [15] An electron is in the + state when measured in the direction $S_\theta = S_z\cos\theta + S_x\sin\theta$, so that $S_\theta\left|+_\theta\right\rangle = +\frac{1}{2}\hbar\left|+_\theta\right\rangle$. However, the angle θ is uncertain. In each part, it is probably a good idea to check at each step that the trace comes out correctly.

(a) [3] Suppose the angle is $\theta = \pm\frac{1}{3}\pi$, with equal probability for each angle. What is the state operator in the conventional $\left|\pm_z\right\rangle$ basis?

We have, from a variety of sources, the states in terms of this basis, which is

$$\left|+_\theta\right\rangle = \cos\left(\tfrac{1}{2}\theta\right)\left|+\right\rangle + \sin\left(\tfrac{1}{2}\theta\right)\left|-\right\rangle$$

The state vector is then simply taken by averaging the results for the two angles, so

$$\rho = \frac{1}{2}\sum_{\theta=\pm\frac{\pi}{3}}\left|+_\theta\right\rangle\left\langle+_\theta\right| = \frac{1}{2}\left[\begin{pmatrix}\cos\frac{\pi}{6}\\ \sin\frac{\pi}{6}\end{pmatrix}\begin{pmatrix}\cos\frac{\pi}{6} & \sin\frac{\pi}{6}\end{pmatrix} + \begin{pmatrix}\cos\frac{\pi}{6}\\ -\sin\frac{\pi}{6}\end{pmatrix}\begin{pmatrix}\cos\frac{\pi}{6} & -\sin\frac{\pi}{6}\end{pmatrix}\right]$$

$$= \frac{1}{2}\left[\begin{pmatrix}\cos^2\frac{\pi}{6} & \cos\frac{\pi}{6}\sin\frac{\pi}{6}\\ \cos\frac{\pi}{6}\sin\frac{\pi}{6} & \sin^2\frac{\pi}{6}\end{pmatrix} + \begin{pmatrix}\cos^2\frac{\pi}{6} & -\cos\frac{\pi}{6}\sin\frac{\pi}{6}\\ -\cos\frac{\pi}{6}\sin\frac{\pi}{6} & \sin^2\frac{\pi}{6}\end{pmatrix}\right] = \begin{pmatrix}\cos^2\frac{\pi}{6} & 0\\ 0 & \sin^2\frac{\pi}{6}\end{pmatrix} = \begin{pmatrix}\frac{3}{4} & 0\\ 0 & \frac{1}{4}\end{pmatrix}.$$

The result has trace one, so it's probably right.

(b) [4] Suppose the angle θ is randomly distributed in the range $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, with all angles equally likely. What is the state operator in the conventional $\left|\pm_z\right\rangle$ basis?

Instead of adding two angles, we need to integrate over all angles, and divide by the range of angles, which is π, so we have

$$\rho = \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left|+_\theta\right\rangle\left\langle+_\theta\right| d\theta = \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\begin{pmatrix}\cos\left(\frac{1}{2}\theta\right)\\ \sin\left(\frac{1}{2}\theta\right)\end{pmatrix}\begin{pmatrix}\cos\left(\frac{1}{2}\theta\right) & \sin\left(\frac{1}{2}\theta\right)\end{pmatrix}d\theta$$

$$= \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\begin{pmatrix}\cos^2\left(\frac{1}{2}\theta\right) & \cos\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\theta\right)\\ \cos\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\theta\right) & \sin^2\left(\frac{1}{2}\theta\right)\end{pmatrix}d\theta = \frac{1}{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\begin{pmatrix}1+\cos\theta & \sin\theta\\ \sin\theta & 1-\cos\theta\end{pmatrix}d\theta$$

$$= \frac{1}{2\pi}\begin{pmatrix}\theta+\sin\theta & -\cos\theta\\ -\cos\theta & \theta-\sin\theta\end{pmatrix}_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{2\pi}\left[\begin{pmatrix}\frac{\pi}{2}+1 & 0\\ 0 & \frac{\pi}{2}-1\end{pmatrix} - \begin{pmatrix}-\frac{\pi}{2}-1 & 0\\ 0 & -\frac{\pi}{2}+1\end{pmatrix}\right] = \begin{pmatrix}\frac{1}{2}+\frac{1}{\pi} & 0\\ 0 & \frac{1}{2}-\frac{1}{\pi}\end{pmatrix}.$$

Once again, the trace is one, so it's probably correct.

(c) [4] Suppose the angle θ is randomly distributed in the range $-\pi < \theta < \pi$, with all angles equally likely. What is the state operator in the conventional $\left|\pm_z\right\rangle$ basis?

This is identical to the previous part, except the range is twice as big and of course the limits of integration change, so we have

$$\rho = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|+_\theta\right\rangle\left\langle +_\theta\right| d\theta = \cdots = \frac{1}{4\pi}\int_{-\pi}^{\pi}\begin{pmatrix} 1+\cos\theta & \sin\theta \\ \sin\theta & 1-\cos\theta \end{pmatrix} d\theta = \frac{1}{4\pi}\begin{pmatrix} \theta+\sin\theta & -\cos\theta \\ -\cos\theta & \theta-\sin\theta \end{pmatrix}_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi}\left[\begin{pmatrix} \pi & -1 \\ -1 & \pi \end{pmatrix} - \begin{pmatrix} -\pi & -1 \\ -1 & -\pi \end{pmatrix}\right] = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

So this is a completely unpolarized state operator. It again has trace one.

(d) [4] In each of the cases listed above, what is the expectation value of S_z?

The expectation value can be found via

$$\langle S_z \rangle = \mathrm{Tr}\left(\rho S_z\right) = \tfrac{1}{2}\hbar\,\mathrm{Tr}\left(\rho\sigma_z\right)$$

In every case, this trace is easy to work out.

$$\langle S_z \rangle_a = \tfrac{1}{2}\hbar\,\mathrm{Tr}\left[\begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \tfrac{1}{2}\hbar\left(\tfrac{3}{4}-\tfrac{1}{4}\right) = \tfrac{1}{4}\hbar,$$

$$\langle S_z \rangle_b = \tfrac{1}{2}\hbar\,\mathrm{Tr}\left[\begin{pmatrix} \frac{1}{2}+\frac{1}{\pi} & 0 \\ 0 & \frac{1}{2}-\frac{1}{\pi} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \tfrac{1}{2}\hbar\left(\tfrac{1}{2}+\tfrac{1}{\pi}-\tfrac{1}{2}+\tfrac{1}{\pi}\right) = \tfrac{1}{\pi}\hbar$$

$$\langle S_z \rangle_b = \tfrac{1}{2}\hbar\,\mathrm{Tr}\left[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \tfrac{1}{2}\hbar\left(\tfrac{1}{2}-\tfrac{1}{2}\right) = 0.$$

Perhaps not surprisingly, the first two cases have a positive expectation value, while the third vanishes. This is because in the first two cases, though the spin is random, it's definitely at an angle that is closer to $+z$ than $-z$, but in the third it is equally likely at all angles.